УДК 624.04:531.391.3

Формулы для расчета спектра частот собственных колебаний балочной фермы с произвольным числом панелей

Кирсанов Михаил Николаевич
НИУ «МЭИ»
профессор

Тиньков Дмитрий Владимирович
НИУ «МЭИ»
Аспирант

Аннотация
Рассмотрены малые колебания системы грузов, расположенных в узлах верхнего прямолинейного пояса статически определимой фермы. Масса самой фермы не учитывается. Методом индукции с привлечением системы компьютерной математики Maple выводятся формулы для элементов матрицы податливости, собственные числа которой определяют частоты колебаний. Искомая матрица составляется в виде суммы трех бисимметричных матриц, одна из которых единичная, разреженная нулями. Формулы для элементов матриц получаются тройной индукцией.
Ключевые слова: ферма, малые колебания, частоты, индукция, Maple

Formulas for calculating the frequency spectrum of natural oscillations of a beam truss with an arbitrary number of panels

Kirsanov Mikhail Nikolaevich
NRU «MPEI»
Professor

Tinkov Dmitriy Vladimirovich
NRU «MPEI»
Postgraduate

Abstract
Small oscillations of the cargo system located in the nodes of the upper straight belt of a statically identifiable truss are considered. The mass of the truss itself is not taken into account. The induction method using the Maple computer mathematics system derives formulas for the elements of the compliance matrix, whose eigenvalues determine the oscillation frequencies. The required matrix is composed of the sum of three bisymmetric matrices, one of which is the unit matrix, sparse with zeros. Formulas for matrix elements are obtained by triple induction.
Keywords: truss, small oscillations, frequencies, induction, Maple

Introduction to the formulation of the problem. Analysis of truss natural frequencies is one of the most important engineering tasks, especially in relation to the structures used in transport construction and mechanical engineering. Truss oscillations arise both during the movement of goods, for example, the movement of transport, and during the dynamic excitation of oscillations from the action of moving parts of instruments and equipment fixed on the truss. At the heart of the analysis of any type of oscillations is information about the natural frequencies. Usually, eigenfrequencies are determined numerically [1-5], based on any specialized packages. An alternative to this approach is analytical research. For regular systems, we apply the induction method, worked out using the Maple computer mathematics system on deflection problems of flat [6-8] and spatial [9-11] trusses, in which the analytical dependencies of the solution were determined not only on the size of the truss and the load, but also on the number of panels. As a rule, a regular truss scheme contains one or two parameters defining the complexity of the structure, for example, the number of panels in a span or in supporting elements. The task of finding the natural frequencies of a system with many degrees of freedom contains at least three parameters. This is due to the fact that the frequencies are associated with the eigenvalues of the matrix, therefore, when obtaining formulas for the elements of the matrix, it is necessary to perform induction three times: by column number, row number and number of panels. Previously, a similar problem was solved for a truss with a triangular lattice [12,13]. In [14], the concentrated mass method was used to solve the problem of natural oscillations of a regular truss. An overview of analytical solutions in problems of deflection of flat regular trusses is given in [15].

Truss. Equations and calculation of forces. Consider a symmetrical truss with a triangular lattice and pillars containing 2n panels in the span (Fig. 1).

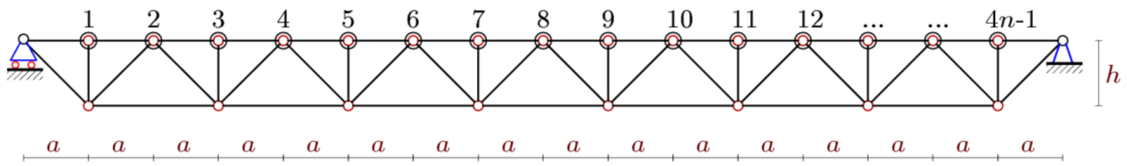

Figure 1 — Truss with n = 4

We write the differential equation of oscillations

$$[M_n]\ddot{\overline{Y}}+[D_n]\overline{Y}=0, \quad (1)$$

Here $\overline{Y}$ is a vector composed of vertical movements of loads. Its length is equal to the number of degrees of freedom $4n-1$. The dots denote the time derivative, $\ddot{\overline{Y}}$ — the acceleration vector, $[D_n]$ — the stiffness matrix of the system, $[M_n]$ — the inertia matrix. In the assumption that the masses of cargo are the same this diagonal. We reduce the problem to the problem of the eigenvalues of the compliance matrix $[B_n]$ inverse to the stiffness matrix. Its elements are calculated using the Maxwell-Mohr formula:

$$b_{i,j}=\sum_{\nu=1}^{n_s-3} S_\nu^{(i)} S_\nu^{(j)} l_\nu /(EF). \qquad (2)$$

Denotes: $S_\nu^{(i)}$ — force from the action of unit force in node i in the bar with a number ν, l_ν – bar length, EF — bar stiffness, $n_s=12n+2$ — number of rods, including three support rods. Rod sections are the same. The sum is made for all truss rods, except for three supporting ones.

Multiply (1) by $[B_n]$:

$$[B_n]\bar{Y}=\lambda\bar{Y},$$

where

$$\lambda=1/(m\omega^2) \qquad (3)$$

— own number , ω — natural frequency of oscillation.

The calculation using the Maple program used and described in [6-11] begins with solving the problem of forces in the rods included in (2). To do this, you must enter the data on the coordinates of the nodes (hinges). The origin of coordinates is located in the left support of the truss:

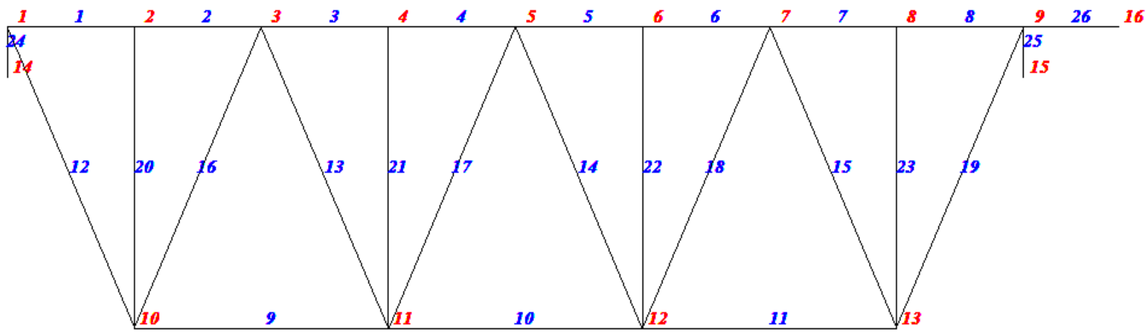

Figure 2 — The numbering of nodes and rods with n = 2

Here is a fragment of the coordinate input program:

```
> for i  to 4*n+1 do        # upper belt
 >  x[i]:=a*(i-1):y[i]:=h:
> end:
> for i  to 2*n do          # lower belt
>  x[i+4*n+1]:=2*a*(i-1)+a:y[i+4*n+1]:=0:
> end:
```

The lattice structure is defined by special conditional vectors corresponding to the rods and containing the number of nodes at the ends as coordinates. The beginning and end of the rod are chosen arbitrarily and do not affect the force and sign of the force. We have the following vectors:

```
> for i to 4*n do
N[i]:=[i,i+1];                  # upper belt
> end:
> for i to 2*n-1 do
> N[i+4*n]:=[i+4*n+1,i+4*n+2];#  lower belt
```

```
> end:

> for i to 2*n do              # lattice
>  N[i+6*n-1]:=[2*i-1,i+4*n+1];
>  N[i+8*n-1]:=[2*i+1,i+4*n+1];
>  N[i+10*n-1]:=[2*i,i+4*n+1];
> end:
```

The cosines of the forces and loads are introduced into the matrix of the system of equilibrium equations. Here it is a single vertical force applied to the nodes of the upper belt

```
> for i to 4*n-1 do P[i][2*i+2]:=1: od:
```

The vector P[i], where $i = 1, \dots, 4n-1$ is the number of the mass, in even elements it contains the values of the vertical forces applied to the node, and in odd elements they are horizontal. Here is the node with the number $i + 1$. The solution of the system of equations is obtained by the method of the inverse matrix G1: = 1 / G, where G is the matrix of the system

Induction. The calculation of the trusses for different n shows that the desired matrix consists of the sum of three matrices

$$[B_n]=([A_n]a^3+n[C_n]c^3+4n^2[H_n]h^3)/(4n^2h^2EF), \tag{4}$$

where $c=\sqrt{a^2+h^2}$. The main task is to derive formulas for the elements of these matrices. For $n = 1$, we have matrices

$$[A_1]=\begin{bmatrix}7&8&5\\8&12&8\\5&8&7\end{bmatrix},\ [C_1]=\begin{bmatrix}3&2&1\\2&4&2\\1&2&3\end{bmatrix},\ [H_1]=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

At $n = 2$

$$[A_2]=\begin{bmatrix}70&112&130&128&110&80&42\\112&200&240&240&208&152&80\\130&240&310&320&282&208&110\\128&240&320&352&320&240&128\\110&208&282&320&310&240&130\\80&152&208&240&240&200&112\\42&80&110&128&130&112&70\end{bmatrix},$$

$$[C_2]=\begin{bmatrix}7&6&5&4&3&2&1\\6&12&10&8&6&4&2\\5&10&15&12&9&6&3\\4&8&12&16&12&8&4\\3&6&9&12&15&10&5\\2&4&6&8&10&12&6\\1&2&3&4&5&6&7\end{bmatrix},\quad [H_2]=\begin{bmatrix}1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\end{bmatrix}.$$

Matrix is bisymmetric [16, 17]. This means that to obtain all the elements of a matrix, it is sufficient to derive formulas for the base triangle — the elements of the upper half of the matrix between its main and secondary diagonals, including the diagonals. For example, for a matrix $[C_n]$ with n = 2, these are nonzero elements.

$$\begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 12 & 10 & 8 & 6 & 4 & 0 \\ 0 & 0 & 15 & 12 & 9 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Further, a reflection on the diagonals can be obtained and the entire matrix:

```
> for j to 4*n-1 do
>  for i from j+1 to 4*n-1 do
       C[i,j]:=C[j,i]:
    od:
  od:
> for j to 4*n-1 do
>  for i from j+1 to 4*n-1 do
      C[i,4*n-j]:=C[4*n-i,j]:
    od:
  od:
```

The most time consuming process was the derivation of the elements of the matrix. At the first stage, for a fixed number of panels *n*, it is necessary to derive its common term in each row of the matrix within the base triangle. For small values of *n*, the result cannot be obtained. The rgf_findrecur operator to obtain a recurrent equation, which is satisfied by the elements, lacks the length of the sequence. In this problem, this length is equal to eight. When *n* = 5, we have recurrent equations of the form

$$Z_j = 4Z_{j-1} - 6Z_{j-2} + 4Z_{j-3} - Z_{j-4},$$

where Z_j – is a string element. The following expressions are common terms of strings derived from solving a recurrent equation using the rsolve operator

$$a_{1,j} = (5j^3 - 300j^2 + 4000j)/3,$$

$$a_{2,j+1} = 2(5j^3 - 285j^2 + 3430j + 3420)/3,$$

$$a_{3,j+2} = (5j^3 - 270j^2 + 2900j + 6120),$$

$$a_{4,j+3} = 4(5j^3 - 255j^2 + 2410j + 8160)/3,$$

...

At the second stage, a summary is made in rows (j)

$$a_{i,i+j-1} = (5j^3 - (315-15i)j^2 - \alpha_1 j + \alpha_0)i/3,$$

$$\alpha_1 = 20i^2 - 630i + 4610, \qquad (5)$$

$$\alpha_0 = 10i^3 - 420i^2 + 4610i - 4200.$$

For example, to obtain the coefficient α_1, it was necessary to generalize the sequence 4000, 3430, 2900, 2410, 1960, 1550 according to an equation $\alpha_{1,i} = 3\alpha_{1,i-1} - 3\alpha_{1,i-2} + \alpha_{1,i-3}$ which gives the operator rgf_findrecur.

Similarly, when n = 6, we have the following lines

$$a_{1,j} = (6j^3 - 432j^2 + 6912j)/3,$$
$$a_{2,j+1} = 2(6j^3 - 414j^2 + 6084j + 6072)/3,$$
$$a_{3,j+2} = (6j^3 - 396j^2 + 5304j + 11088),$$
$$a_{4,j+3} = 4(6j^3 - 378j^2 + 4572j + 15120)/3,$$
$$\ldots$$

We summarize these expressions in rows (j)

$$a_{i,i+j-1} = (6j^3 - (450 - 18i)j^2 - \alpha_1 j + \alpha_0)i/3,$$
$$\alpha_1 = 24i^2 - 900i + 7788, \qquad (6)$$
$$\alpha_0 = 12i^3 - 600i^2 + 7788i - 7200.$$

Omitting the intermediate actions, we give the first rows of the compliance matrix with n = 7

$$a_{i,i+j-1} = (7j^3 - (609 - 21i)j^2 - \alpha_1 j + \alpha_0)i/3,$$
$$\alpha_1 = 28i^2 - 1218i + 12166, \qquad (7)$$
$$\alpha_0 = 14i^3 - 812i^2 + 12166i - 11368.$$

At the final stage, formulas (5) - (7) are summarized by the number of panels n. For example, the sequence 4200, 7200, 11368, 16896, 23976, 32800, 43560, 56448 with n = 5, 6, ... , 12 free members in the coefficient has a common term $\gamma_0 = 8n^2(1+4n)$, which is a solution of the equation of the eighth order

$$\gamma_{0,n} = 4\gamma_{0,n-1} - 6\gamma_{0,n-2} + 4\gamma_{0,n-3} - \gamma_{0,n-4}.$$

As a result, we have expressions for the elements of the basis triangle of the matrix $[A_n]$ for any value of n:

$$a_{i,i+j-1} = (nj^3 - (\alpha_2 - 3ni)j^2 + \alpha_1 j + \alpha_0)i/3,$$
$$\alpha_2 = 3n(1+4n),\ \alpha_1 = 4ni^2 - \beta_1 i + \beta_0,$$
$$\beta_1 = 6n(1+4n),\ \beta_0 = 2n(16n^2 + 12n + 1),$$
$$\alpha_0 = 2ni^3 - \gamma_2 i^2 + \gamma_1 i - \gamma_0,$$
$$\gamma_2 = 4n(1+4n),\ \gamma_1 = \beta_0,\ \gamma_0 = 8n^2(1+4n).$$

Much simpler, but also in three stages of generalization, the elements of the matrix are obtained $[C_n]$

$$c_{i,i+j-1} = (4n - i - j + 1)i,\ i = 1,\ldots,2n,\ \ j = 1,\ldots,4n - 2i - 1.$$

To obtain a diagonal unit sparse matrix $[H_n]$, the induction method is not required. Its elements do not depend on n:

$h_{i,i} = (1-(-1)^i)/2,\ i = 1,...,4n-1.$

Frequencies. Knowing the eigenvalues of the (3) compliance matrix, we find the oscillation frequencies. Using the Eigenvalues operator of the Maple system, we calculate the matrix eigenvalues for a truss with n = 1. The problem has an exact solution:

$$\lambda_1 = (a^3 + c^3 + 2h^3)/(2EFh^2), \tag{8}$$

$$\lambda_{2,3} = \left(6a^3 + 2c^3 + h^3 \pm \sqrt{h^6 + 32a^6 + 16a^3c^3 + 2c^6}\right)/(2EFh^2).$$

The same eigenvalues have matrices for other values of n. Hence, the spectrum of the natural frequencies of the truss with two panels in the span (n = 1) is included as a subset in the frequency spectra of the trusses with an arbitrary number n. If we write out rather cumbersome formulas for the eigenvalues of the truss matrix with n = 2, then it turns out that this spectrum is included in all the spectra of trusses with even n. A more extensive analysis with trusses for various, sufficiently large n, suggests that in general there is a property of embedding the spectra of trusses with the number of panels $n = k_1$ and $n = k_2$ into the spectrum of a truss with $n = k_1k_2$.

This assumption requires proof in the general case, but it can be to some extent verified numerically. Consider an example of a truss with height h = 4 m, a panel length a = 3 m and stiffness of rods $EF = 2,0\cdot 10^4$ kN. In the nodes of the upper belt are mass m = 200 kg. Computing with the help of the ***Eigenvalues*** operator of the Maple system, the eigenvalues of the matrix for a truss with a different number of panels n, we get 4n – 1 frequencies of the vibration spectrum. We note the oscillation frequencies of each truss (n = 1, 3, 4, ..., 9) with dots on the curve corresponding to this truss (Fig. 3).

The abscissa axis of the graph shows the numbers of k frequencies in the spectrum, ordered in ascending order. Analyzing the graphs, we note the similarity of curves constructed for trusses of the same height and with the same length a and different numbers of panels. In addition, it is clear that the frequency spectra are always divided into two parts with a sharp jump in the middle of the spectrum. The average frequency corresponds to the eigenvalue (8). At lower altitudes (h = 4m), the noted frequency jump in the middle of the spectrum almost disappears (Fig. 4).

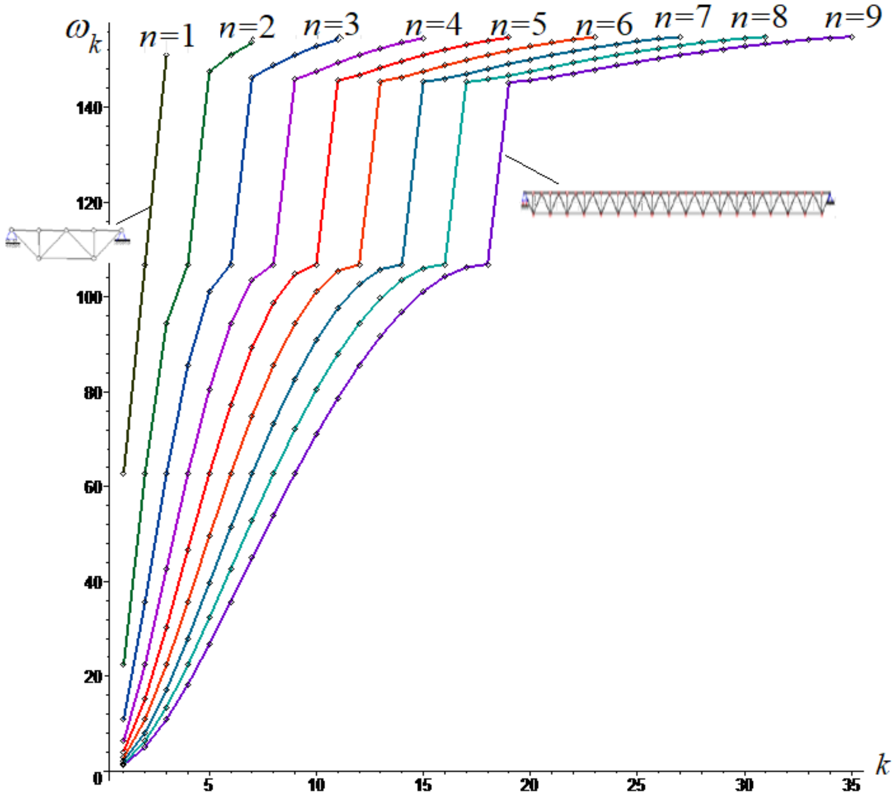

Figure 3 — Frequency spectra (rad/c) for trusses with different number of panels, h = 4 m

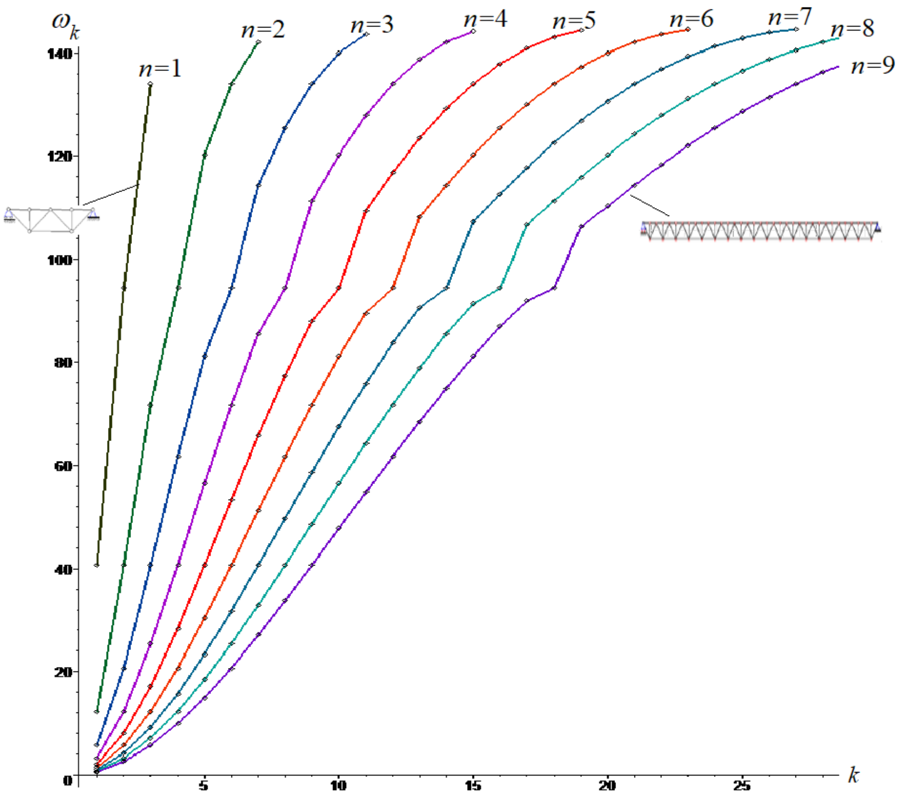

Figure 4 — Frequency spectra (rad/c) for trusses with different number of panels, h = 2 m

The frequency $\omega_1 = 1/\sqrt{m\lambda_1}$ dependence graph, where the eigenvalue λ_1 is calculated by formula (8), constructed for the same values as the graphs in Figure 3, indicates the presence of a maximum of the average frequency depending on the height *h* (Fig. 5).

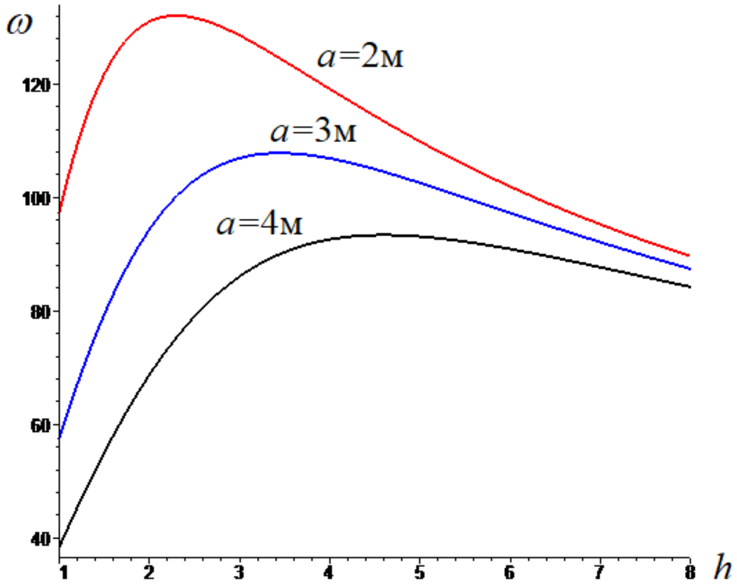

Figure 5 — The dependence of the average frequency of the height

The value of the height at which the frequency ω_1 is maximum has a simple dependence on the panel length *a*

$$h^* = a((1/3)(36+21\sqrt{3})^{2/3}-1)/(36+21\sqrt{3})^{1/3} \approx 1{,}149a.$$

The maximum frequency (Fig. 3, 4) is almost independent of the number of panels.

References

1. Kilikevičius A., Bačinskas D., Jurevičius M., Kilikevičiene K., Fursenko A., Jakaitis J., Toločka, E. Field testing and dynamic analysis of old continuous truss steel bridge // Baltic Journal of Road & Bridge Engineering. 2018. 13(1). https://doi.org/10.3846/bjrbe.2018.394
2. Bačinskas D., Kamaitis I. Z., Kilikevičius A. A sensor instrumentation method for dynamic monitoring of railway bridges// Journal of Vibroengineering. 2013. 15(1). pp. 176-184.
3. Bačinskas D., Turla V., Kilikevičius A., Ragauskas, P., Jurevièius M. Dynamic testing of railway truss bridge // Journal of vibroengineering, 2014. 16(6), pp. 2649-2657. Retrieved from https://www.miestai.net/forumas/
4. Camara A., Nguyen K., Ruiz-Teran A. M. Stafford P. J. Serviceability limit state of vibrations in under-deck cable-stayed bridges accounting for vehicle-structure interaction // Engineering Structures. (2014). 61(1). pp. 61-72.

https://doi.org/10.1016/j.engstruct.2013.12.030

5. Zambon I., Vidovic A., Strauss A., Matos J., Amado J. Comparison of stochastic prediction models based on visual inspections of bridge decks. Journal of Civil Engineering and Management. 2017. (23) 5. pp. 553-561.
6. Кирсанов М.Н. Аналитический расчет прогиба распорной фермы с произвольным числом панелей Механизация строительства. 2017. Т. 78. № 3. С. 26-29.
7. Кирсанов М.Н. Индуктивный анализ деформации арочной фермы // International Journal for Computational Civil and Structural Engineering. 2018. Т. 14. № 1. С. 64-70.
8. Кирсанов М.Н. Аналитическое исследование деформаций плоской фермы арочного типа // Вестник государственного университета морского и речного флота им. адмирала С.О. Макарова. 2015. № 3 (31). С. 42-48.
9. Kirsanov M.N. Analytical calculation, marginal and comparative analysis of a flat girder // Scientific Herald of the Voronezh State University of Architecture and Civil Engineering. Construction and Architecture. 2016. № 1 (29). С. 84-105.
10. Кирсанов М.Н. Аналитический расчет прогиба пространственного прямоугольного покрытия // Вестник МГСУ. 2018. Т. 13. № 5 (116). С. 579-586.
11. Кирсанов М.Н. Учет строительного подъема в аналитическом расчете пространственной балочной фермы // Известия Московского государственного технического университета МАМИ. 2014. Т. 4. № 2 (20). С. 36-39.
12. Канатова М.И. Частотное уравнение и анализ колебаний плоской балочной фермы // Trends in Applied Mechanics and Mechatronics. М: Инфра-М. 2015. Т. 1. С. 31-34.
13. Ахмедова Е.Р., Канатова М.И. Собственные частоты колебаний плоской балочной фермы регулярной структуры // Наука и образование в XXI веке: сборник научных трудов по материалам Международной научно-практической конференции 31 октября 2014 г. в 17 частях. Часть 11. Тамбов: ООО «Консалтинговая компания Юком», 31 октября 2014. С. 17-18.
14. Рыбаков Л. С., Мишустин И. В. Применение метода сосредоточенных масс к анализу собственных упругих колебаний одной регулярной ферменной структуры // Механика композиционных материалов и конструкций. 1999. Т. 5. №. 4. С. 51-64.
15. Осадченко Н.В. Аналитические решения задач о прогибе плоских ферм арочного типа // Строительная механика и конструкции. 2018. Т.1. №16. С. 12–33.
16. Кирсанов М.Н., Черепанов С.П. О некоторых бисимметричных матрицах регулярного типа // В сборнике: Международная школа «Математическое моделирование фундаментальных объектов и явлений в системах компьютерной математики - KAZCAS-2016». Международная научно-практическая конференция «Информационные технологии в образовании

и науке - ИТОН-2016» Труды школы и материалы конференции. Под общей редакцией Ю.Г. Игнатьева. 2016. С. 53-54.

17. Vorobiev O., Kirsanov M., Cherepanov S. About some bisymmetric matrix of regular type // В сборнике: Наука и образование в XXI веке сборник научных трудов по материалам Международной научно-практической конференции: в 34 частях. 2013. С. 9-10.