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ANALYTICAL SOLUTION OF A SPATIAL CONSOLE GIRDER FREQUENCY OSCILLATIONS PROBLEM

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Statement of the problem. A scheme of a hinged-rod cantilever structure and two algorithms for deriving the dependence of its fundamental frequency on the number of panels are proposed. The deflection of the girder and the force in the support rods is analyzed.
Materials and methods. The values of the force in the rods and the reactions of the supports are found in an analytical form using the node-cutting method in the Maple symbolic mathematics system. The structure is fixed to the wall by three supports, one of which is a spherical hinge, the other supports are a cylindrical hinge and a rack. The rigidity of the structure is calculated using the Maxwell — Mohr formula. The frequency of free oscillations is calculated using the Dunkerley method and the modified Rayleigh's method. Generalization of the solutions to an arbitrary number of panels of the structure is performed by the induction method.
Results. Assuming that the mass of the structure is uniformly distributed over its nodes, and the rods are weightless, formulas are derived for the dependence of the fundamental frequency of truss oscillations on its dimensions and the number of panels. Vertical oscillations of the nodes are considered. The coefficients in the calculation formula are obtained in the computer mathematics system and have the form of polynomials in the number of panels. The results of analytical solutions are compared with numerical ones. It is shown that for a certain number of panels, the modified Rayleigh method gives several times higher accuracy than the Dunkerley method. The error of the calculation formulas depends on the number of panels and the height of the truss.
Conclusions. The proposed scheme of the hinged-rod console allows for a simple in form and fairly accurate analytical solution suitable for analyzing large-scale structures. The algorithm used to derive the calculation formula can be applied to other regular statically determinate structures.

Keywords: console, spatial girder, deflection, induction, Maple, natural frequency, Dunkerley method, Rayleigh method, analytical solution.

Introduction. Along with strength and deformations, the first natural frequency of free vibrations of engineering structures is one of the major characteristics of structures. In engineering practice the calculation of the fundamental natural frequency of building structures is normally performed numerically using the finite element method [5, 15]. At the same time calculations of large-scale structures containing a large number of elements in their structure have the property of accumulating rounding errors causing the accuracy being undermined. For the first time, the problem of the existence of schemes of statically determinate regular girders had been addressed by R. G. Hutchinson, N. A. Fleck, F. W. Zok, R. M. Latture, M. R. Begley [6, 7, 20]. In [16] while solving problems of calculating and optimizing girders based on the idea of a genetic algorithm, a mechanism is set forth aimed at reducing the complexity of optimization problems of structural size and topology by means of converting complex search spaces into simpler fuzzy domains. In an approach to expanding the application of the continuum model to girder-like lattice truss structures with geometric nonlinearity is presented. Geometric nonlinearity in girder structures is considered by dividing it into separate independent structures and introducing a coordinate system of joint rotation in order to decompose the movement of the parts into that of a rigid body and pure deformations. The static deflection of a flat lattice truss with two fixed articulated supports in analytical form was obtained in [19]. Inductive methods of deducing the analytical dependence of the fundamental frequency on the order of the system are employed to calculate statically definable regular structures. A tool for developing such solutions is typically a kind of a computational mathematics system, e. g., *Maple* or *Mathematica* [8, 21]. This was how static solutions for a Molodechno type girder [9], flat multi-grid girder [10], spatial L-shaped girder [12], for flat regular arched girders [14, 17] were obtained. In [11] the Maxwell — Mohr formula is used to identify the movements of the frame nodes with a double lattice. An analytical solution for deflection of an externally statically indeterminate frame in the *Maple* system was obtained in [1] using an inductive generalization of a series of solutions for regular girders of a variety of orders. A lower approximate estimate of the first natural frequency of the natural oscillations of a flat girder is obtained in [18]. A variant of the Dunkerley method, where the sum is replaced by its average value for the sake of simplification, is used in [13] to calculate the fundamental natural frequency of a flat girder with two fixed hinged supports. The spectrum of all frequencies of a family of regular girders of a variety of orders is also numerically designed and analyzed. The deflection of the four-pitched roof girder in an analytical form using the *Maple* system is analyzed in [2].

In this study the problem of deflection and the first frequency of natural oscillations of a statically definable cantilever-type spatial girder (Fig. 1, 2) consisting of four flat trusses with a diagonal grid is addressed. The girder has the height h_1+h_2 and length na. The end face structure is mounted on three supports. Support *A* is spherical, *B* is cylindrical, and at the top *C* the truss is mounted on a horizontal rod. In total there are $n_s=12(n+1)$ rods and $K=4(n+1)$ internal hinges in the girder.

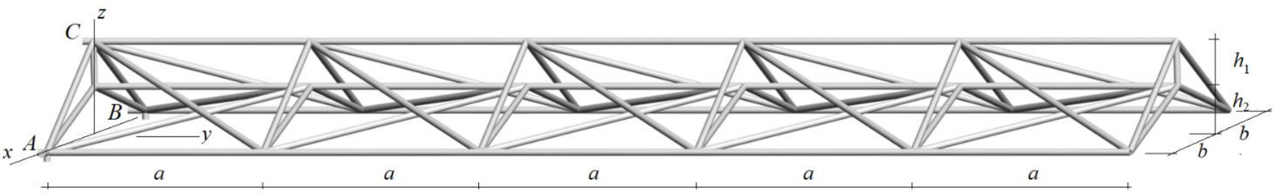

Fig. 1. Girder, $n = 5$

The task is to identify the analytical dependences of the deflection and the first oscillation frequency of the girder on its size and the number of panels.

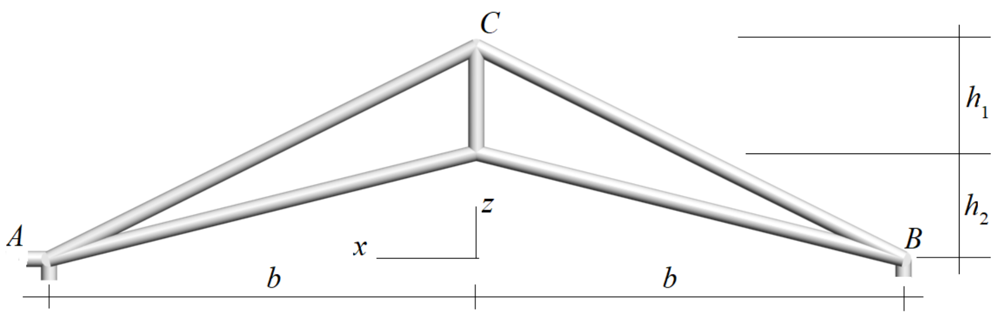

Fig. 2. Girder in the plane *x-z*

1. Calculating the force. The force in the rods of the structure under an external load is identified analytically based on the equilibrium equations of its nodes. The guiding cosines of the force that makes up the matrix of the system are calculated using the coordinates of the nodes as well as the vectors that determine the structure of the connections of the rods into nodes [9, 11]. The program is written in the symbolic mathematics *Maple* system [1, 24]. The coordinates of the nodes in the accepted coordinate system take the form:

$$\begin{aligned}
&x_i = b,\ y_i = a(i-1),\ z_i = 0,\\
&x_{i+n+1} = 0,\ y_{i+n+1} = y_i,\ z_{i+n+1} = h_2,\\
&x_{i+2n+2} = -b,\ y_{i+2n+2} = y_i,\ z_{i+2n+2} = 0,\\
&x_{i+3n+3} = 0,\ y_{i+3n+3} = y_i,\ z_{i+3n+3} = h_1 + h_2,\ i = 1,..,n+1.
\end{aligned}$$

The order of connecting the rods to the nodes is determined by the special lists $T_i,\ i = 1,...,n_s,$ containing the end numbers of the corresponding rods. The horizontal longitudinal rods on the edges of the girder are coded, e. g., as follows:

$$T_{i+(j-1)n} = [i+(j-1)(n+1), i+(j-1)(n+1)+1],\ i = 1,...,n,\ j = 1,2,3.$$

The coefficients of the equilibrium equations of the nodes in the projections on the coordinate axis are calculated based on the data on the coordinates of the nodes and the order of connecting the rods to the nodes. The system of linear equations of node equilibrium is written in the vector form: $\mathbf{AS} = \mathbf{B}$ where $\mathbf{A}$ is the matrix containing the guiding cosines of the force in the rods; $\mathbf{S}$ is the vector of the forces in the rods. Six reactions of the supports are included in the vector $\mathbf{S}$. The components of the vector $\mathbf{B}$ consist of projections of external loads on the nodes. Three consecutive elements of the vector $\mathbf{B}$ correspond to each node *i*. The external loads in the projection onto the axis *x* are entered into the elements such as B_{3i-2} and into the elements B_{3i-1} onto the axis *y*. The vertical loads are written into the elements such as B_{3i}.

Under an even vertical (Fig. 3) load loads on the non-zero elements of the vector $\mathbf{B}$ take the form: $B_{3i} = -P,\ i = 1,...,n_s$.

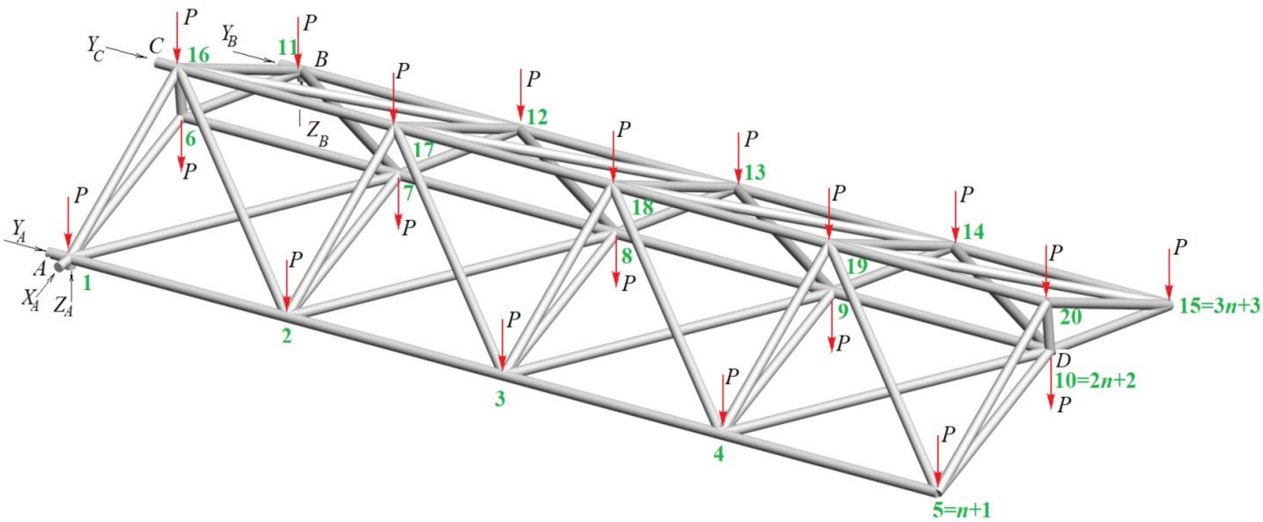

Fig. 3. Numbers of the nodes and load on the girder, *n* = 4

2. Deflection of the console. Calculation of the vertical displacement of the end of the console *D* (node number 2(n+1)) is performed according to the Maxwell — Mohr formula:

$$\Delta_n = \sum_{j=1}^{n_s} \frac{S_j s_j l_j}{EF},$$

where *EF* is the longitudinal rigidity of the straining girder; l_j is the length of the *j*-th rod and S_j is the force in it caused by a load; s_j is the force in the same rod depends on a single vertical force applied to node *D*. The six support rods are assumed to be deformable and the Maxwell — Mohr sum includes. The lengths of the horizontal support rods are conventionally assumed to be equal to *a*, vertical to *h*. These lengths determine the stiffness of the supports.

Sequential calculation of the deflection of the girder at *n* = 1, …, 5 and $h_1 = h_2 = h/2$ yields the following results:

$$\Delta_1 = P(16a^3 + 4c^3 + 24d^3 + 14h^3 + 3f^3)/(2h^2EF),$$
$$\Delta_2 = P(434a^3 + 48c^3 + 176d^3 + 84h^3 + 19f^3)/(8h^2EF),$$
$$\Delta_3 = P(746a^3 + 48c^3 + 136d^3 + 56h^3 + 13f^3)/(4h^2EF),$$
$$\Delta_4 = P(3760a^3 + 160c^3 + 384d^3 + 140h^3 + 33f^3)/(8h^2EF),$$
$$\Delta_5 = P(1975a^3 + 60c^3 + 128d^3 + 42h^3 + 10f^3)/(2h^2EF),...,$$

where $$c = \sqrt{a^2 + b^2 + h^2}, \quad d = \sqrt{b^2 + h^2}, \quad f = \sqrt{4b^2 + h^2},$$

In the general case:

$$\Delta_n = P\left(C_1a^3 + C_2c^3 + C_3d^3 + C_4h^3 + C_5f^3\right)/\left(h^2EF\right). \tag{1}$$

In order to identify the common terms of the sequences of the coefficients in these expressions for degrees of size, it is necessary to continue this sequence to at least ten. In this case the operators *rgf_findrecur* and *rsolve* of the *Maple* system yield the following values:

$$\begin{aligned} C_1 &= n(n+1)(35n^2 + 137n + 20)/48, \\ C_2 &= n(n+1), \\ C_3 &= n^2 + 7n + 4, \\ C_4 &= (7n+7)/2, \\ C_5 &= (7n+5)/8. \end{aligned} \tag{2}$$

Solution (1) with coefficients (2) has a quadratic asymptote in relation to the number of panels. Using the *Maple* methods with the *limit*(*D*1/*n*^2,*n* = *infinity*) operator for the relative value $\Delta' = EF\Delta_n/(L_0P_{sum})$ where $P_{sum} = 4P(n+1)$ is the total vertical load onto all the nodes of the girder, a limit $\lim\limits_{n\to\infty} \Delta'/n^2 = 35a^2/(192h^2)$ can be found.

At the same time as identifying the deflection, the calculation of the forces in the rods also provides analytical expressions for the reactions of the supports:

$$Z_A = Z_B = 2P\left(n+2\right),$$
$$Y_A = Y_B = Pa\left(n+1\right)n/h,$$
$$Y_C = -2Pa\left(n+1\right)n/h.$$

The method of induction is not required for these formulas. The reactions are derived from the six equations of equilibrium of the girder as a rigid whole.

3. Formula for the first natural oscillation frequency of the girder. The Dunkerley method. Assuming that the mass of the farm is concentrated in its nodes, which oscillate along the *z* axis, the number of degrees of freedom of the system is *K*. The simplest analytical estimate of the first frequency is specified by means of the Dunkerley method [12—14]:

$$\omega_D^{-2} = \sum_{i=1}^{K} \omega_i^{-2}, \tag{3}$$

where ω_i is the oscillation frequency of the mass m in the nodes of the girder.

The equation of vertical mass fluctuations is

$$m\ddot{z}_i + d_i z_i = 0,$$

where z_i is the horizontal displacement of the mass; $\ddot{z}_i$ is the acceleration; d_i is the coefficient of rigidity. The partial frequency of the node is

$$\omega_i = \sqrt{d_i / m}\,.$$

The coefficient of rigidity is the inverse of the coefficient of compliance δ_i calculated using the Maxwell — Mohr formula:

$$\delta_i = 1/d_i = \sum_{\alpha=1}^{n_s} \left(S_\alpha^{(i)}\right)^2 l_\alpha / (EF).$$

Here $S_\alpha^{(i)}$ is the force in the rod numbered α under a single horizontal force applied to the node numbered i where the mass is. Based on (3),

$$\omega_D^{-2} = m\sum_{i=1}^{K} \delta_i = m\Delta_n. \tag{4}$$

Similarly to the deflection calculation algorithm, the sequential calculation of the sum Δ_n for girders of different orders yields the following expressions:

$$\Delta_1 = (24a^3 + 12c^3 + 36d^3 + 24h^3 + 5f^3 + g^3)/(2h^2EF),$$
$$\Delta_2 = (517a^3 + 154c^3 + 298d^3 + 138h^3 + 37f^3 + 15g^3)/(8h^2EF),$$
$$\Delta_3 = (1184a^3 + 236c^3 + 380d^3 + 136h^3 + 45f^3 + 24g^3)/(6h^2EF),$$
$$\Delta_4 = (3737a^3 + 530c^3 + 770d^3 + 225h^3 + 89f^3 + 55g^3)/(8h^2EF),$$
$$\Delta_5 = (9472a^3 + 1000c^3 + 1360d^3 + 336h^3 + 155f^3 + 105g^3)/(10h^2EF),...,$$

where

$$g = \sqrt{4a^2 + 4b^2 + h^2}\,.$$

In the general case this multiplier has the form:

$$\Delta_n = \left(C_1a^3 + C_2c^3 + C_3d^3 + C_4h^3 + C_5f^3 + C_6g^3\right)/\left(h^2EF\right). \tag{5}$$

The coefficients here are identified using *Maple* as solutions of homogeneous linear recurrent equations:

$$\begin{aligned} &C_1 = (27n-7)(n+1)(3n^2+10n+23)/120, \\ &C_2 = (n+1)(41n-5)/12, \\ &C_3 = (41n+67)(n+1)/12, \\ &C_4 = (n+1)(11n+1)/(2n), \\ &C_5 = (3n^2+8n+9)/8, \\ &C_6 = (n+1)(3n-1)/8. \end{aligned} \tag{6}$$

Thus the calculation formula for the lower estimate of the first frequency by means of the Dunkerley method has the form:

$$\omega_D = h\sqrt{\frac{EF}{m(C_1a^3+C_2c^3+C_3d^3+C_4h^3+C_5f^3+C_6g^3)}}. \tag{7}$$

4. Rayleigh method. The formula for estimating the first frequency of vibrations of the girder from above using the Rayleigh energy method has the form [8]:

$$\omega_R^2 = \sum_{i=1}^{K} u_i \Big/ \sum_{i=1}^{K} m u_i^2. \tag{8}$$

Here u_i is the displacement of the mass in the node with the number i when a uniformly distributed unit nodal load acts on the girder. If the sum $\sum_{i=1}^{K} u_i$ is replaced by its average $u_* K/2$ where u_* is the deflection of the node with the maximum value and perform the same for the denominator (8), following the reductions the formula for an approximate estimate of the first frequency will take the simple form:

$$\omega_R^2 = 1/(m u_*). \tag{9}$$

Thus in order to calculate u_* , solution (1) at P = 1 with the coefficients (2) can be used:

$$\Delta = \left(C_1a^3+C_2c^3+C_3d^3+C_4h^3+C_5f^3\right)/\left(h^2EF\right).$$

Hence the calculation formula for the first frequency using the simplified Rayleigh method will take the form:

$$\omega_R = h\sqrt{\frac{EF}{m(C_1a^3+C_2c^3+C_3d^3+C_4h^3+C_5f^3)}}. \tag{10}$$

The errors of the analytical Dunkerley method using formula (7) with coefficients (6) and the solution using the Rayleigh method (10) with coefficients (2) can be compared with the lowest frequency of the spectrum obtained numerically based on the analysis of the girder oscillation problem with K degrees of freedom. The numerical solution is also found in the *Maple* system using the *Eigenvalues* operator to identify the eigenvalues of a matrix from the *LinearAlgebra* package of the *Maple* system.

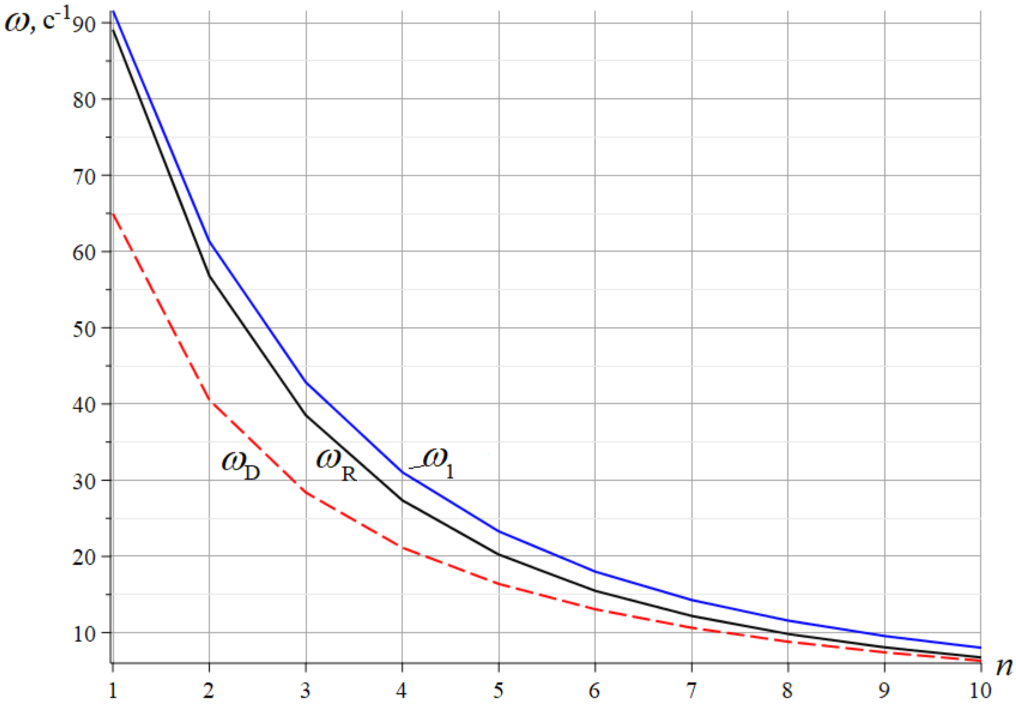

Fig. 4. Dependence on the number of panels of the first natural frequency ω_D of the fluctuations of the girder according to the Dunkerley method and the first frequency ω_1 of the spectrum found numerically at h = 3 m

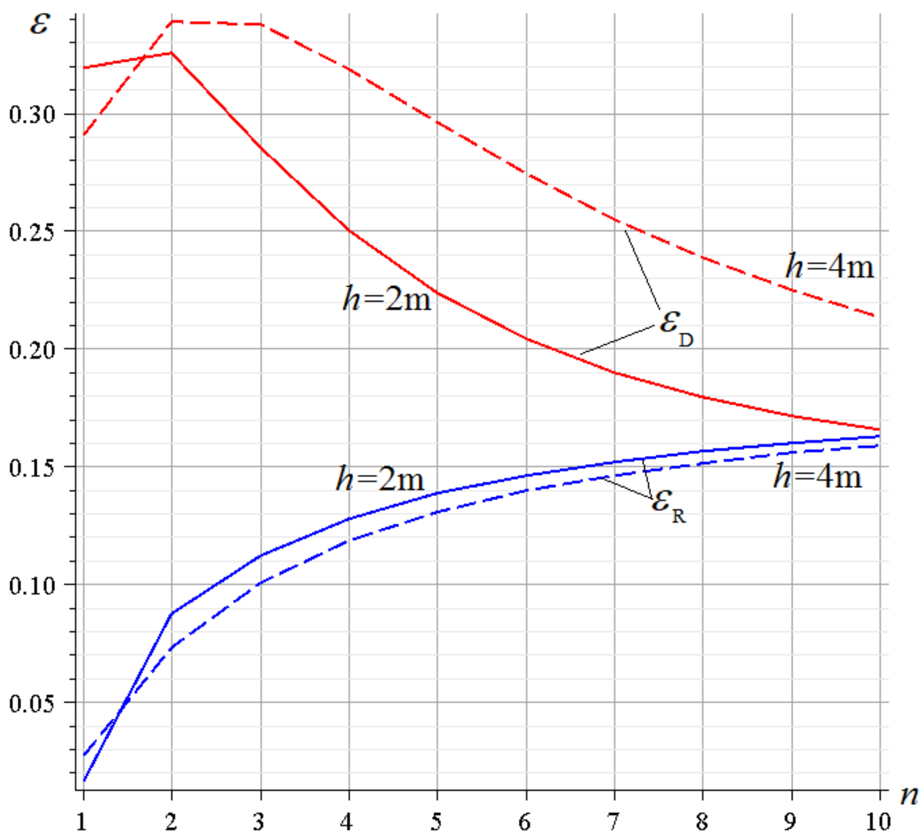

Fig. 5. Dependence of the errors of the analytical estimates of the first frequency on the number of panels

The dependence curves of the first frequency of the numerical solution ω_1, analytical ω_D using formula (7) and ω_R using formula (10) are compared in graph 4 at $E = 2{,}1 \cdot 10^5$ MPa, $F = 9$ см^2, $m = 200$ кг, a = 3 m, h = 2 m and b = 1 m. As the number of panels rises, so does the accuracy of analytical estimates. This is shown in graph 5 of the relative errors

$$\varepsilon_R = |\omega_1 - \omega_R| / \omega_1 \text{ and } \varepsilon_D = |\omega_1 - \omega_D| / \omega_1$$

of the number of panels. For a small number of panels, the suggested version of the Rayleigh method yields an error significantly lower than the Dunkerley method, while the error of the Rayleigh method increases all the time as does the order of the girder n, and the error of the Dunkerley method decreases. Starting from a certain value of n, the errors are reversed. At the same time their values are relatively small — about 20 %. The errors also depend on the height of the girder h. It should be noted that the original Rayleigh method according to formula (8) yields a greater accuracy than the above two methods, however, the solution turns out to be cumbersome.

Conclusions. Formulas for the dependence of the first oscillation frequency on the number of panels and size of a structure are derived in two ways for the spatial scheme of a statically determinate regular cantilever rod structure. The frequency calculation using the modified Rayleigh's method makes use of an analytical solution to the static problem of deflection of the end of the console under a distributed nodal load. A comparison with the frequency solution obtained by means of the conventional numerical method with the calculation of the eigenvalues of the matrix proves the results to be accurate.

The suggested console scheme can be employed in civil and industrial construction and the formulas in preliminary design calculations.

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