SINGULAR POINTS OF THE CREEP DEFORMATION AND BUCKLING OF A COLUMN

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Abstract – A criterion of creep buckling based on singular points of the deformation process is suggested. The theory of transient creep with power law strain hardening is examined. A moment of the history of the deformation process is said to be a singular point if the initial value of speed, acceleration or some time-derivative of deflection corresponds to infinity of deflection at this moment. It is shown that the problem of singular points reduces to an eigenvalue problem. The first point of the sequence coincides with the Rabotnov and Shesterikov criterion.

1. INTRODUCTION

Constructions tend to buckling at compressive load. In the condition of the elastic or plastic deformation, this phenomenon is bound up with bifurcation of state or process of loading. The bifurcation approach for estimation of the critical time is based on a linear equation and is independent of an initial imperfection value. Thus the first criterions of creep buckling were based on particular hypothesis by analogy with elastic (Gerard [1]) or plastic (Shanley [2]) instability. Comprehensive reviews of development in this field were given by Hoff [3], Kurshin [4], Arutyunian, Drozdov, and Kolmanovsky [5].

The first approach to the creep buckling problem, similar to the bifurcation method, was carried out by Rabotnov and Shesterikov [6], where the deflection of a construction and its speed were analyzed. Later Klyushnikov [7,8] has generalized this approach to a high time-derivative. The sequence obtained (pseudo bifurcation points) turns out to be increasing, and its first point corresponds to the Rabotnov and Shesterikov criterion. This fact stimulates the advance of the bifurcation theory because Rabotnov- Shesterikov's point underestimates the creep collapse time. Meanwhile the pseudo bifurcation sequence was unbounded and Klyushnikov has advanced an assumption that the system buckles after it safely has passed some points of pseudo bifurcation.

The mathematical formalism used in the derivation of pseudo bifurcation points is based on the hypothesis of independence of the time-derivatives of the deflection. This assumption simplifies the solution to the problem, especially in a three-dimensional case. In this paper, another definition of pseudo bifurcation points with due account of the linkage between time-derivatives of the deflection is suggested. The new points are named singular. The algorithm to find such points for the creep of a centrally loaded column is given.

2. DEFINITION OF SINGULAR POINTS

Let us consider creep of a simply supported column of length l compressed by the force T. A uniaxially nonlinear constitutive law will be employed here in the form [9]

$$\dot{p}p^{\alpha} = f(\sigma), \tag{1}$$

where $p = \varepsilon - \sigma/E$ is the creep strain, a dot indicates differentiation with respect to time $\dot{p} = dp/dt$, α is a material constant, $f(\sigma)$ is a stress function (power, exponential, etc.).

Some disturbance whose nature is not investigated here results in the displacement of an initially straight column and increments in the stress $\Delta\sigma$ and creep strain Δp. Linearization of the constitutive law (1) gives

$$p^{\alpha}\Delta\dot{p} + \alpha\dot{p}p^{\alpha-1}\Delta p = f'\Delta\sigma, \tag{2}$$

where $f' = df/d\sigma$. Due to Bernoulli's hypothesis and static equilibrium equation we have

$$\int_A \Delta\varepsilon dA = J\Delta v,_{yy}\,, \int_A \Delta\sigma dA = -T\Delta v, \tag{3}$$

wherein z is the coordinate in the plane of bending, y is the axial coordinate, $0 < y < l$; $\Delta v(y)$ is the deflection at any point y, $\Delta v,_{yy}$ is the second-order derivative of the deflection with respect to y; A and J are the cross sectional area and second moment of inertia, respectively. Substituting eqs. (3) into $\Delta p = \Delta\varepsilon - \Delta\sigma/E$ yields

$$\int_A \Delta pzdA = T\Delta v/E + J\Delta v,_{yy}\,. \tag{4}$$

Let us rewrite (2) in terms of the deflection and its speed with due of eqs.(3) and (4). After the multiplication (2) by z and integration over the cross-sectional area using eqn(1) and the fact that the quantities without symbol Δ are independent of z, we have

$$Jp\Delta\dot{v},_{yy} + Tp\Delta\dot{v}/E + \alpha\dot{p}(T\Delta v/E + J\Delta v,_{yy}) = -Tp\dot{p}\Delta vf'/f. \tag{5}$$

The deflection is taken in the form $\Delta v = U_0 \sin\mu y$ to satisfy boundary conditions $\Delta v = 0$ at $y = 0$ and $y = l$; $\mu = mp/l$, $(m = 1, 2..)$. Hence from eqn (5) we obtain $\Delta\dot{v} = (\dot{p}/p)U_1 \sin\mu y$.

Let us introduce the quantity ξ as a measure of pure compression process of a theoretically straight column before buckling

$$\xi = p(f'/f)E\sigma/(\sigma_0 - \sigma), \tag{6}$$

where $\sigma_0 = EJ\mu^2/A$ is the critical stress of an elastic column. Substituting the form of the deflection U_0 into (5), we have the equation for the amplitudes U_0 and U_1

$$(\alpha - \xi)U_0 + U_1 = 0. \tag{7}$$

Thus, the increments of the deflection U_0 and its speed U_1 are connected by eqn (7). Vanishing of the coefficient $\alpha - \xi$ at U_0 corresponds to the Rabotnov and Shesterikov criterion [6] and defines a special point characterized by the parameter $\xi_1 = \alpha$. Critical sense of this state occurs from the fact that before

ξ_1 the deflection of the column decreases in the following motion, but the same disturbance applied after ξ_1 gives an increasing deflection. Later [7] this point was named as the pseudo bifurcation point of the zeroth order (PB0 after the order of time-derivative that is ambiguous in critical point) and other points of the higher orders (PBN) were carried out.

Note that the point ξ_1 has another interpretation. The speed U_1 given at the point ξ_1 according to eqn(7) leads to the infinity of the deflection U_0. Just in such an interpretation the singular points will be defined here. The point ξ we shall name the singular point of the first order.

Reasoning along similar lines, we can propose the existence of the second singular point ξ_2 that is characterized by decreasing of the given speed before ξ_2 and increasing after it. In another interpretation this point corresponds to the infinity of the speed of the deflection when the acceleration is assigned in ξ_2. It is necessary to have the equation containing the acceleration of the deflection. If we raise the order of the constitutive law

$$\ddot{p}p^{\alpha} + \alpha\dot{p}^2 p^{\alpha-1} = f'\dot{\sigma}, \tag{8}$$

the acceleration will appear in the equation for amplitudes. In the case of constant loading ($\dot{T} = \dot{\sigma}A = 0$) for small increments, we have from eqn (8)

$$\Delta\ddot{p}p^{\alpha} + 2\alpha\dot{p}p^{\alpha-1}\Delta\dot{p} + \alpha[\ddot{p}p + (\alpha-1)\dot{p}^2]p^{\alpha-2}\Delta p = f'\Delta\dot{\sigma}. \tag{9}$$

At the condition $\dot{\sigma} = 0$, eqn(8) leads the identity

$$\ddot{p} = -\alpha\dot{p}^2/p \tag{10}$$

correct for the quantities of a primary process.

Substituting this relation into eqn (9) and repeating the above procedure with use of eqs.(3) and (4), and similar equations for the speeds of the stress and creep strain increments, we have

$$-\alpha U_0 + (2\alpha - \xi)U_1 + U_2 = 0. \tag{11}$$

An alternative way of developing this equation is time differentiating of eqn (2.7).

The amplitude of the acceleration we introduce in the form

$$\Delta\ddot{v} = (\dot{p}/p)^2 U_2 \sin\mu y. \tag{12}$$

The two equations (8) and (11) form a system for three quantities U_0, U_1, and U_2. Taking the amplitude of the acceleration U_2 to be known and rearranging it to the right side of the system, we can write it in the form

$$\begin{bmatrix} \alpha-\xi & 1 \\ -\alpha & 2\alpha-\xi \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} U_2. \tag{13}$$

The system has the solution

$$U_0 = U_2/B_2, \quad U_1 = -(\alpha - \xi)U_2/B_2, \tag{14}$$

where $B_2 = \xi^2 - 3\alpha\xi + \alpha(1 + 2\alpha)$ is the determinant of the matrix in eqn(13).

Let us define the second order singular point ξ_2 as such a moment of the creep process, when the initial acceleration causes the infinity of the initial deflection and speed. On the bases of solutions (14), ξ_2 is a root of the equation $B_2 = 0$

$$\xi = (3\alpha \pm (\alpha^2 - 4\alpha)^{1/2})/2. \tag{15}$$

Note that the difference between a number of variables U_k appeared in system (13) and its order is equal to the order of the higher time-derivative in the assumed constitutive law, and agree with the number of independent initial conditions on the perturbation motion. In the problem considered, this number is 1.

One can expand the system (13) by rising the order of the constitutive law. Let us derive it for an arbitrary order. To display a generality of its construction, it is sufficient to restrict by the fourth order. Let us differentiate eqn(8) with respect to time

$$p^{(3)}p^\alpha + 3\alpha\ddot{p}\dot{p}p^{\alpha-1} + \alpha(\alpha - 1)\dot{p}^3 p^{\alpha-2} = f''\dot{\sigma}^2 + f'\ddot{\sigma}. \tag{16}$$

In view of the condition $\dot{\sigma} = 0$, eqn (16) leads the identity for the fundamental process

$$p^{(3)} = \dot{p}^3\alpha(2\alpha + 1)/p^2. \tag{17}$$

Equation (16) linearized with help of eqs.(10) and (17) in terms of U_k takes the form

$$\alpha(\alpha + 2)U_0 - 3\alpha U_1 + (3\alpha - \xi)U_2 + U_3 = 0, \tag{18}$$

where U_3 is introduced by the formula similar to (12). In much the same way we, obtain the fourth equation

$$F_4U_0 + 4F_3U_1 + 6F_2U_2 + (4F_1 - \xi)U_3 + U_4 = 0. \tag{19}$$

The functions $F_i = F_i(p)$ are the coefficients in eqs. (8), (11), (18), and (19)

$$F_0 = 1,\ F_1 = \alpha,\ F_2 = -\alpha,\ F_3 = \alpha(\alpha + 2),\ F_4 = -\alpha(\alpha + 2)(2\alpha + 3), ...$$

There is the recurrent formula for F_N

$$F_{N+1} = -[N + \alpha(N - 1)]F_N, \qquad N = 1, 2, 3....$$

We can write the system of eqs. (8), (11),(18) and (19) in the matrix form

$$\begin{bmatrix} F_1 - \xi & 1 & 0 & 0 \\ F_2 & 2F_1 - \xi & 1 & 0 \\ F_3 & 3F_2 & 3F_1 - \xi & 1 \\ F_4 & 4F_3 & 6F_2 & 4F_1 - \xi \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} U_4, \tag{20}$$

There exist four singular points $\xi_1 ... \xi_4$ resulting from the system (20) according to the order of time-derivative of the deflection amplitude U_k which is taken as the known value in the right side of the system. Particularly, taking U_2 to be known in the initial condition of perturbation motion, we have

$$\begin{bmatrix} F_1-\xi & 1 & 0 & 0 \\ F_2 & 2F_1-\xi & 0 & 0 \\ F_3 & 3F_2 & 1 & 0 \\ F_4 & 4F_3 & 4F_1-\xi & 1 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ U_3 \\ U_4 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 3F_1-\xi \\ 6F_2 \end{bmatrix} U_2, \qquad (21)$$

The determinant of this system is $(F_1-\xi)(2F_1-\xi)-F_2 = B_2$ and agrees with that of the system (13). Its vanishing corresponds to the singular point ξ_2. Moreover, from the solution of (21), it is clear that not only U_0 and U_1 tend to the infinity at the moment ξ_2 but the high derivative U_3 also does. Generalizing this fact, we give the following definition of singular points. The N-order singular point is such a moment of creep history at which the initial value of the N-order time-derivative of the deflection amplitude causes the infinity of the initial deflection and its time-derivatives $U_i, (i \neq N)$.

System (20) (i.e. fundamental system) may be rewritten in the form

$$[\mathbf{M} - \xi\mathbf{I}]\bar{\mathbf{U}} = \bar{\mathbf{Z}}.$$

All the components Z_i in the right side are equal to zero except $Z_N = -U_N$; $\mathbf{I}$ is the identity matrix. Elements of matrix $\mathbf{M}$ are

$$m_{ij} = C^i_{j-1}F_{i-j+1}, \quad C^i_j = i!/[j!(i-j)!], \quad i,j = 1,2...N,$$

where $F_i = 0$ if $i < 0$. The rule of taking summation over a repeating index is not accepted.

The problem of the singular points reduces to the eigenvalue problem of the matrix $\mathbf{M}$. We write the secular polynomials to solve it [10]

$$\begin{aligned} &B_1 = \xi - \alpha, \quad B_2 = \xi^2 - 3\xi\alpha + \alpha(2\alpha+1), \\ &B_3 = \xi^3 - 6\alpha\xi^2 + \alpha(4+11\alpha)\xi - \alpha(2\alpha+1)(3\alpha+2), \\ &B_4 = \xi^4 - 10\alpha\xi^3 + 5\alpha(2+7\alpha)\xi^2 - \\ &\quad 5\alpha(2\alpha+1)(5\alpha+2)\xi + \alpha(2\alpha+1)(3\alpha+2)(4\alpha+3). \end{aligned}$$

The recurrent relation is true

$$B_N = \xi B_{N-1} - \sum_{i=0}^{N-1} B_i C^N_i F_{N-i}, \qquad B_0 = 1, N = 1,2,3...$$

The roots ξ_N of the polynomials B_N $(N < 7)$ against α are plotted on Fig.1. The odd polynomials always have at last one root for any α but the even ones have a solution in particular range of α. For example B_2 has the solution (15) if $\alpha > 4$, and polynomials B_4 and B_6 have any solutions if $\alpha > 2.4$ and 1.9, respectively, what may be found out from the numerical solution.

Let us compare the results obtained with the results of the previous approaches. For most of them, the predicted critical creep strain may be represented in the form $p = \gamma(\sigma_0 - \sigma)/E$. The dimensionless coefficient γ is shown in the comparative table.

The idea of Kurshin's approach [4] to creep buckling problem is close to the Rabotnov and Shesterikov's one, to the fist-order pseudo bifurcation point PB1, and second-order singular point ξ_2. Its foundation connected with vanishing of the acceleration of the perturbation at the critical moment $\xi = \xi^*$. It was noted that if a disturbance is applied to a column before ξ^* (we use ξ as a time scale) when the creep strains are rather small, then the lateral deflection increases sluggishly. But if the motion begins after ξ^*, the deflection is accelerated. Both of two similar criterions [4] depending on initial conditions of perturbation motion are presented in table in separate rows.

According to eqn(6) for the N-order singular point, we have

$$p = \xi_N \frac{(\sigma_0 - \sigma) f}{f' E \sigma}, \tag{22}$$

where ξ_N are the roots of the polynomials B_N.

3. COMPARISON WITH THE EXPERIMENTAL DATA

The solution of the singular point problem of the compressed column is plotted on the axis of the dimensionless load $\omega = \sigma/\sigma_0$ and relative strain $\varepsilon_0 = pE/\sigma_0$ (Fig.2).

Let us consider the experiment by Chapman, J.C., Erickson, B. and Hoff N.J. [11]. Creep buckling tests were conducted on aluminum alloy 2024-T4 columns in axial compression at $260C^\circ$. The data for specimens are as follows: $l = 96.5\text{mm}, A = 12.7 \times 6.35\text{mm}^2$. The theoretical value of the elastic critical stress $\sigma_0 = 208.8MPa$ (it does not even approach in the experiment) is used in calculations. The results are fitted by the creep constitutive equation (1) with $f(\sigma) = C\sigma^n,\ n = 9,\ \alpha = 2.4,\ C = 7.9 \times 10^{-31} MPa^{-9}\text{min}^{-1},\ E = 5.87 \times 10^4 MPa$. The quantity ξ has the following values: $\xi_1 = 2.40$, $\xi_3 = 4.55$, $\xi_5 = 5.77$, $\xi_6 = 6.10$, $\xi_7 = 6.38$ (we use only the first roots of the polynomials). Corresponding lines $\varepsilon_0 = \xi_N(1 - \omega)/n$ pass through $\omega = 1,\ \varepsilon_0 = 0$. The experimental data are marked off by circles. It is evident that the experiment is in close agreement with the curves ξ_5, ξ_6 and ξ_7.

4. CONCLUSION

The theory suggested is designed only for analysis of perfect systems buckling. This is the distinctive feature of the singular points theory. We do not overlook the familiar point of view based on the idea of unavoidable initial imperfections that grew with time under the applied load and then cause buckling. It is reasonable to expect that there exist systems (or will exist in future) with negligibly small imperfection for which the theory of initial imperfection will fail. In this case the singular points theory is extremely useful.

Criterion	γ		
	$f(\sigma)$	$f = C\sigma^n$	$\alpha = 2.4$ $n = 9$
Gerard [1]	1	1	1.000
Shanley[2]	$(1+\alpha)f/(f'\sigma)$	$(1+\alpha)/n$	0.378
Kurshin[4] 1.	$2\alpha f/(f'\sigma)$	$2\alpha/n$	0.533
2.	$(1+2\alpha)f/(f'\sigma)$	$(1+2\alpha)/n$	0.644
Rabotnov, Shesterikov[6]	$\alpha f/(f'\sigma)$	α/n	0.267
Klyushnikov[7]	$(N+1)\alpha f/(f'\sigma)$	$(N+1)\alpha/n$	
1.		α/n	0.267
2.		$2\alpha/n$	0.533
3.		$3\alpha/n$	0.800
4.		$4\alpha/n$	1.067
Singular points	$\xi_N f/(f'\sigma)$	ξ_N/n	
1.		ξ_1/n	0.267
2.		ξ_2/n	-
3.		ξ_3/n	0.505
4.		ξ_4/n	-
5.		ξ_5/n	0.641

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Fig 1. Singular points ξ of the order from 1 to 7 vs the material constant α.

Fig 2. Comparison of the creep buckling dimensionless load $\omega = \sigma/\sigma_0$ of a column corresponding to the singular points of the order N from 1 to 7 with the experimental data from reference [11].

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