S.Targ *Theoretical Mechanics. A Short Course.* Mir Publishers. Moscow 1988

Theorem of the Change in the Kinetic Energy of a System

§147. Kinetic Energy of a System

The kinetic energy of a system is defined as a scalar quantity equal to the arithmetical sum of the kinetic energies of all the particles of the system:

$$T = \sum_k \frac{m_k v_k^2}{2} \tag{41}$$

Kinetic energy is a characteristic of both the translational and rotational motion of a system, which is why the theorem of the change in kinetic energy is so frequently used in problem solutions. The main difference between and the previously introduced characteristics Q and K_0 is that kinetic energy is a scalar quantity, and essentially a positive one. It, therefore, does not depend on the directions of the absolute motions of parts of a system and does not characterise the changes in these directions.

Another important point should be noted. Internal forces act on the parts of a system in mutually opposite directions. For this reason, as we have seen, they do not change the vector parameters Q and K_0. But if, under the action of internal forces, the speeds of the particles of a system change, the quantity will change too. Consequently, the kinetic energy of a system differs further from the quantities Q and K_0 in that it is affected by the action of both external and internal forces.

If a system consists of several bodies, its kinetic energy is, evidently, equal to the sum of the kinetic energies of all the bodies:

$$T = \sum_k T_k.$$

Let us develop the equations for computing the kinetic energy of a body in different types of motion.

(1) Translational Motion. In this case all the points of a body have the same velocity, which is equal to the velocity of the centre of mass.

Therefore, for any point k we have $v_h = v_c$, and Eq. (41) gives:

$$T_{trans} = \sum \frac{m_k v_C^2}{2} = \frac{1}{2}\left(\sum m_k\right) v_C^2$$

or

$$T_{trans} = \frac{1}{2} M v_C^2. \tag{42}$$

Thus, in translational motion, the kinetic energy of a body is equal to half the product of the body's mass and the square of the velocity of the centre of mass. The value of does not depend on the direction of motion.

(2) **Rotational Motion.** The velocity of any point of a body rotating about an axis Oz (see Fig. 310) is $v_k = \omega h_k$, where h_k is the distance of the point from the axis of rotation and is the angular velocity of the body. Substituting

this expression into Eq. (41) and taking the common multipliers outside of the parentheses, we obtain:

$$T_{rotation} = \sum \frac{m_k \omega^2 h_k^2}{2} = \frac{1}{2}\left(\sum m_k h_k^2\right)\omega^2.$$

The term in the parentheses is the moment of inertia of the body with respect to axis $z4$. Thus, we finally obtain:

$$T_{rotation} = \frac{1}{2} J_z \omega^2. \tag{43}$$

i.e., in rotational motion, the kinetic energy of a body is equal to half the product of the body's moment of inertia with respect to the axis of rotation and the square of its angular velocity. The value of does not depend on the direction of the rotation.

(3) **Plane Motion**[1]. In plane motion, the velocities of all the points of a body are at any instant directed as if the body were rotating about an axis perpendicular to the plane of motion and passing through the instantaneous centre of zero velocity P (Fig. 316). Hence, by Eq. (43),

$$T_{plane} = \frac{1}{2} J_p \omega^2, \tag{43$'$}$$

where J_P is the moment of inertia of the body with respect to the instantaneous axis of rotation, and ω is the angular velocity of the body.

The quantity J_P in Eq. (43') is variable, as the position of the centre P continuously changes with the motion of the body. Let us introduce instead of J_P a constant moment of inertia J_c with respect to an axis through the centre of mass C of the body. By the parallel axis theorem (§132), $J_P = J_c + Md^2$, where $d = PC$. Substituting this expression for J_P into Eq. (43') and taking into account that point P is the instantaneous centre of zero velocity and therefore $\omega d = \omega PC = v_C$, where v_c is the velocity of the centre of mass, we obtain finally:

$$T_{plane} = \frac{1}{2} M v_C^2 + \frac{1}{2} J_p \omega^2, \tag{44}$$

Thus, in plane motion, the kinetic energy of a body is equal to the kinetic energy of translation of the centre of mass plus the kinetic energy of rotation relative to the centre of mass.

(4)* **The Most General Motion of a Body** . Taking the centre of mass as the pole (Fig. 317), the most general motion of a body is a combination of a translation with the velocity v_c of the pole and

[1] This case can be developed as a particular case of the most general motion of a rigid body discussed in the following item.

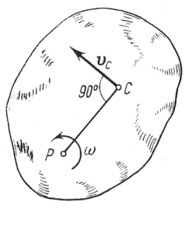

Fig. 316

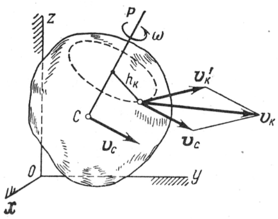

Fig. 317

a rotation about the instantaneous axis CP through the pole (see §88). Then, as shown in the course of kinematics, the velocity v_k of any point of the body is equal to the geometrical sum of the velocity v_c of the pole and the velocity v'_k of the point in its rotation with the body about axis CP:

$$\vec{v}_k = \vec{v}_c + \vec{v}'_k.$$

In magnitude $v'_k = \omega h_k$, where h_k is the distance of the point from axis CP and is the *absolute* angular velocity of the body about that axis. It follows from this that[2]

$$v_k^2 = \vec{v}_k^2 = (\vec{v}_c + \vec{v'}_k)^2 = v_c^2 + {v'}_k^2 + 2\vec{v}_c \cdot \vec{v'}_k.$$

Substituting this expression into Eq. (41) and taking into account that $v'_k = \omega h_k$, we find:

$$T = \frac{1}{2}\left(\sum m_k\right) v_C^2 + \frac{1}{2}\left(\sum m_k h_k^2\right)\omega^2 + \vec{v}_c \cdot \sum m_k \vec{v'}_k,$$

where the common multipliers have been taken outside of the parentheses.

In this equation, the term in the first parentheses gives the mass M of the body and the term in the second parentheses gives the moment of inertia J_{CP} of the body with respect to the instantaneous axis CP. In the last term $\sum m_k \vec{v'}_k = 0$ as it represents the linear momentum of the body in its rotation about axis CP, which passes through the centre of mass (see §138).

Therefore, we finally have:

$$T = \frac{1}{2} M v_c^2 + \frac{1}{2} J_{CP}\omega^2. \tag{45}$$

Thus, in the most general motion of a body, the kinetic energy is equal to the kinetic energy of translation of the centre of mass of the body plus the kinetic energy of rotation about an axis through the centre of mass.

If the pole is taken not in the centre of mass but in another point A such that axis AA' does not pass through the centre of mass, then for this axis $\sum m_k \vec{v'}_k \neq 0$, and we cannot develop an equation of the form (45) (see Problem 136).

[2]From the definition of the scalar product of two vectors (see footnote on p. 300) it follows that $v^2 = \vec{v} \cdot \vec{v} = vv\cos 0 = v^2$, i.e., the scalar square of a vector is equal to the square of its magnitude. This result has been employed here.