Dependence of the first natural frequency of the trussed frame on the number of panels: analytical solution

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Abstract. The inertial properties of a planar model of a hinged supported frame-type truss are modeled by point masses at the nodes. Each mass has two degrees of freedom. The stiffness matrix of a statically determinate structure is determined by the Maxwell-Mohr formula. To find the forces in the truss rods in an analytical form, the method of cutting nodes is used, which is implemented in the Maple computer mathematics system. The lower bound of the first natural frequency is sought by the Dunkerley method. Individual solutions for trusses with a sequentially increasing number of panels are generalized to an arbitrary case using the double induction method according to the number of panels in the girder and the number of panels in the supporting side parts of the truss. The analytical dependence is compared with the numerical solution of the problem on the spectrum of natural frequencies of the structure. It is shown that with an increase in the number of panels, both in the girder and in the side racks, the accuracy of the analytical assessment increases.

INTRODUCTION

Calculation of natural vibration frequencies is one of the main tasks of structural dynamics. The first, lowest frequency is of particular importance for practice. As a rule, the entire spectrum of vibration frequencies of the structure is considered to find it. In general, this problem is solved numerically [1–7]. A simple analytical lower bound for the first frequency can be obtained using the Dunkerley method [8,9]. In this approach, the problem of the eigenvalues of the compliance matrix is replaced by a simple calculation of its trace. This approach is especially effective in tasks where the lowest frequency is noticeably lower than all others. The Dunkerley method is convenient because, unlike Rayleigh's method [10–12], it does not require predicting the vibration mode. Rayleigh's method [13] gives an upper estimate for the first natural frequency. With a successful choice of the vibration mode for the first mode, the accuracy of the Rayleigh method is significantly higher than the Dunkerley method, but its implementation is more complicated.

For regular structures with symmetry and periodicity, the Dunkerley method, in combination with the induction method, cannot only give an analytical solution for the first frequency but also find the dependence of the solution on the order of regularity. In regular trusses, this order is equal to the number of panels. The general problems of regular statically determinate trusses were studied in [14–16]. There are some analytical solutions [17–23] for the deflection of planar regular trusses in the Maple system [24]. Handbook [25] contains analytical dependences of the deflection of various regular statically determinate planar schemes of trusses, arches, consoles, and frames on the number of panels for some types of loads, including schemes that allow kinematic variability for some numbers of panels. An analytical calculation of the frequencies of natural oscillations of planar trusses is contained in [26–28].

METHODS

Scheme of the truss

Consider a planar symmetric truss containing $2n$ panels in the crossbar and m panels in each of the supporting side parts (figure 1). The entire mass of the truss is conventionally concentrated in $N_0 = 4n + 4m - 1$ nodes of the structure, except for two support nodes. The mass of each node is μ. The truss contains $\eta = 8n + 8m + 2$ rods. This number also includes three rods that model the left movable support and the fixed right one. We will take into account both vertical and horizontal movements of nodes. Thus, this task is on the oscillation of the system with $N = 2N_0$ degrees of freedom. In most known analytical solutions for such constructions [27], horizontal mass movements are neglected, and only vertical vibrations are considered.

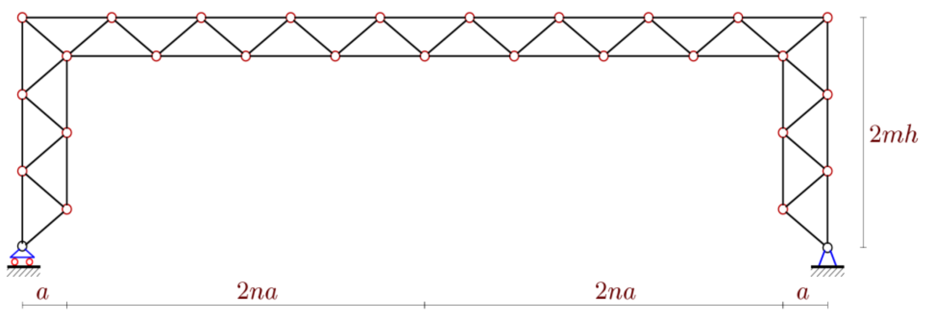

FIGURE 1. Truss, n=4, m=3.

Determination of the forces in the rods

The matrix B_N (the index indicates the number of degrees of freedom) of the pliability of the structure, which is included in the solution of the problem of the natural frequencies of the system, depends on the forces in the rods from the action of unit forces on the nodes in the direction of movement of the nodes and is determined by the Maxwell-Mohr formula:

$$b_{i,j} = \sum_{\alpha=1}^{\eta-3} S_\alpha^{(i)} S_\alpha^{(j)} l_\alpha / (EF), \quad i, j = 1, ..., N, \tag{1}$$

where EF is the stiffness of the rods, $S_\alpha^{(i)}$ is the force in the rod α from the action of a single force at node i; l_α is the length of the rod α. The stiffnesses of all rods are assumed to be the same; therefore, the value EF does not have the α index. Three support rods are assumed to be non-deformable. The compliance matrix of the truss is inverse to the stiffness matrix $D_N = 1 / B_N$.

The forces in the rods of a statically determinate structure are found from the solution of a system of algebraic equations of equilibrium of nodes. The matrix of the system of equations includes the guiding cosines of the forces in the rods. To do this, the nodes and rods are numbered (figure 2), and the coordinates of the nodes and the order of connecting the rods are entered into the program, just as graphs are set in discrete mathematics. To solve the problem, we use a program in the language of the Maple computer mathematics system described in [29]. The algorithm used in this program is easily implemented in other computer mathematics systems (Mathematica, Reduce, Maxima [30]).

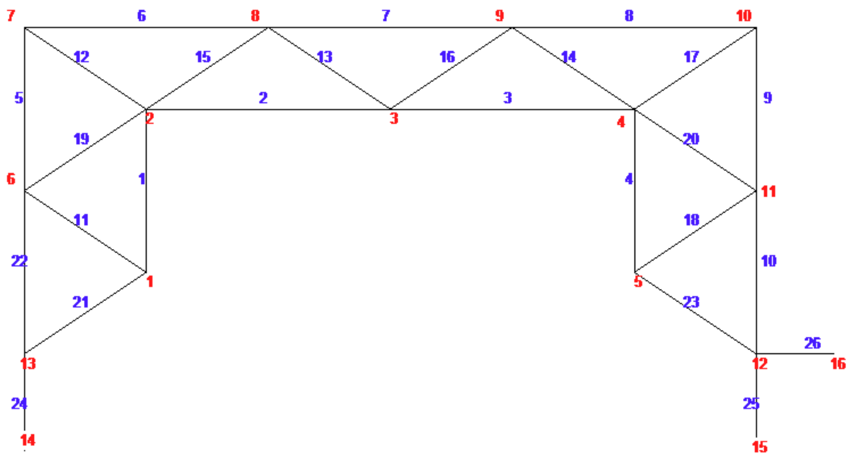

FIGURE 2. Numbering of rods and nodes, n = 1, m=2.

Calculation of the spectrum of natural frequencies

The system of differential equations of vibrations of N_0 loads is written in matrix form:

$$M_N\ddot{U} + D_N U = 0\,, \tag{2}$$

where $U = [u_1, ..., u_N]^T$ are the displacements of the masses. This vector includes both horizontal and vertical mass movements. Denoted by: D_N is the stiffness matrix, M_N is the diagonal inertia matrix of size $N \times N$, and $\ddot{U}$ is the acceleration vector. Since all masses are the same, the inertia matrix is expressed through the unit matrix $M_N = \mu I_N$. If we multiply (2) by B_N on the left, then, taking into account the identity $\ddot{U} = -\omega^2 Y$, the problem is reduced to the problem of the eigenvalues of the matrix B_N: $B_N U = \lambda U,$ where $\lambda = 1/(\mu\omega^2)$ is the eigenvalue of the matrix $\mathbf{B}_N$, ω is the natural frequency of oscillations. The solution to this problem is possible only numerically.

Dunkerley method

We find an analytical expression of the first oscillation frequency using the approximate Dunkerley method. We denote $\omega_{k,v}$ the partial frequency of vibrations of the mass μ at the node k vertically and $\omega_{k,h}$ the frequency of vibrations horizontally. The lower estimate of the first vibration frequency is given by the formula:

$$\omega_D^{-2} = \sum_{k=1}^{N_0} (\omega_{k,v}^{-2} + \omega_{k,h}^{-2}). \tag{3}$$

Equation (2) in the case of oscillations of one mass has a simple scalar form

$$\mu\ddot{u}_k + d_k u_k = 0,\ k = 1, ..., N,$$

where u_k is the displacement of the mass, d_k is the stiffness coefficient. The vibration frequency of an individual mass has the form $\omega_k = \sqrt{d_k / \mu}$. The stiffness d_k is the inverse of the malleability, which is calculated by Maxwell – Mohr's formula:

$$\delta_k = 1/d_k = \sum_{\alpha=1}^{\eta-3} \left(\tilde{S}_\alpha^{(k)}\right)^2 l_\alpha / (EF)$$

According to (3), we have:

$$\omega_D^{-2} = \mu \sum_{k=1}^{N} \delta_k = \mu(\Delta_v + \Delta_h). \tag{4}$$

The sums Δ_v and Δ_h are calculated separately for vertical and horizontal fluctuations, respectively. To obtain the dependence of the solution on the number of panels n in the crossbar and the number of panels m in the supporting parts, double induction is required. To do this, first, for $m = 1$, according to the data of solutions for a sequence of trusses with $n = 1, 2, 3...$, a general formula is obtained, then the same procedure is repeated for $m = 2, 3, 4,...$. A series of solutions obtained for different m is generalized to an arbitrary case. For $m = 1$, we have the following solution for the sum corresponding to the vertical oscillations:

$$\begin{aligned}
&n = 1: \ \Delta_h = 7(79a^3 + 15c^3 + 26h^3)/(18h^2 EF),\\
&n = 2: \ \Delta_h = 11(207a^3 + 15c^3 + 14h^3)/(10h^2 EF),\\
&n = 3: \ \Delta_h = (12103a^3 + 455c^3 + 290h^3)/(14h^2 EF),\\
&n = 4: \ \Delta_h = 19(6681a^3 + 153c^3 + 74h^3)/(54h^2 EF),...
\end{aligned}$$

Here we have introduced a notation for the length of the brace $c = \sqrt{a^2 + h^2}$. Using the operators of the Maple system; we can find common terms of sequences of coefficients in these expressions:

$$m = 1: \qquad \Delta_v = (C_1 a^3 + C_2 c^3 + C_3 h^3)/(h^2 EF), \tag{5}$$

where the coefficients have the form

$$\begin{aligned}
&C_1 = (1024n^5 + 2560n^4 + 2720n^3 + 1520n^2 + 426n + 45)/(90(2n+1)),\\
&C_2 = (32n^3 + 48n^2 + 22n + 3)/(6(2n+1)),\\
&C_3 = (64n^3 + 120n^2 + 74n + 15)/(3(2n+1)^2).
\end{aligned}$$

Similarly, if $m = 2$, we have the following coefficients

$$\begin{aligned}
&C_1 = (1024n^5 + 2560n^4 + 2720n^3 + 2480n^2 + 1026n + 135)/(90(2n+1)),\\
&C_2 = (32n^3 + 48n^2 + 46n + 9)/(6(2n+1)),\\
&C_3 = (128n^3 + 432n^2 + 292n + 66)/(3(2n+1)^2).
\end{aligned}$$

Continuing this process further and finding the coefficients successively for m=3,4,..., we obtain a generalization of the formulas for the coefficients in (5) to an arbitrary number m:

$$C_1 = (1024n^5 + 2560n^4 + 2720n^3 + 80(12m+7)n^2 + 6(100m-29)n - 45 + 90m) / (90(2n+1)),$$
$$C_2 = (32n^3 + 48n^2 + 2(12m-1)n + 6m - 3) / (6(2n+1)),$$
$$C_3 = (64mn^3 + 24(4m^2 + m)n^2 + 2(36m^2 + m)n + 18m^2 - 3m) / (3(2n+1)^2).$$

In the same way, for the sum related to the horizontal mass fluctuations, we have the following expressions for arbitrary numbers m and n

$$\Delta_h = (C_4 a^3 + C_5 c^3 + C_6 h^3) / (a^2 EF),$$
$$C_4 = (32(13n+6)(4n+1)m^3 + 96(16n^3 - 5n^2 - 12n - 3)m^2 -$$
$$- 2(384n^3 + 16n^2 - 212n - 75)m + 288n^3 + 168n^2 + 6n - 9) / (18(2n+1)),$$
$$C_5 = (32(5n+2)m^3 + 48(4n^2 - n - 1)m^2 - 2(48n^2 - 4n - 7)m - 6n + 3) / (6(2n+1)),$$
$$C_6 = (4(56n^2 + 46n + 13)m^4 + 8(32n^3 - 8n^2 - 20n - 7)m^3 - 2(96n^3 + 40n^2 - 4n -$$
$$-13)m^2 + (2n-1)(16n^2 + 16n + 7)m) / (3(2n+1)^2).$$

The final expression of the dependence of the lower bound of the first frequency on the geometric parameters of the structure, including the number of panels in the crossbar and the supporting side trusses, will take the form:

$$\omega_D = \sqrt{\frac{EF}{\mu\left((C_1 a^3 + C_2 c^3 + C_3 h^3) / h^2 + (C_4 a^3 + C_5 c^3 + C_6 h^3) / a^2\right)}}. \tag{6}$$

To check the result, you can perform all the transformations in reverse order: first, perform induction on m, and then on n. It is even better to compare the obtained frequency with the smallest of the entire spectrum of natural oscillation frequencies of the system obtained numerically.

RESULTS AND DISCUSSION

It is possible to solve the oscillation of a system with $N > 3$ degrees of freedom only numerically. Judging by equation (2), the problem is reduced to determining the eigenvalues of the matrix B_N. To do this, we use the *Eigenvalues* operator from the *LinearAlgebra* package of the Maple system. Figure 3 shows the curves of dependence on the number of panels in the crossbar of the first frequency ω_1 of the spectrum obtained numerically and analytically (6) for the value ω_D at m = 4. The elastic modulus of the aluminum is $E = 0.7 \cdot 10^5$ MPa, the mass $\mu =$ 1000 kg, $a =$ 3 m, $h =$ 3 m, the cross-section of the rods $F = 25$ sm^2, are accepted.

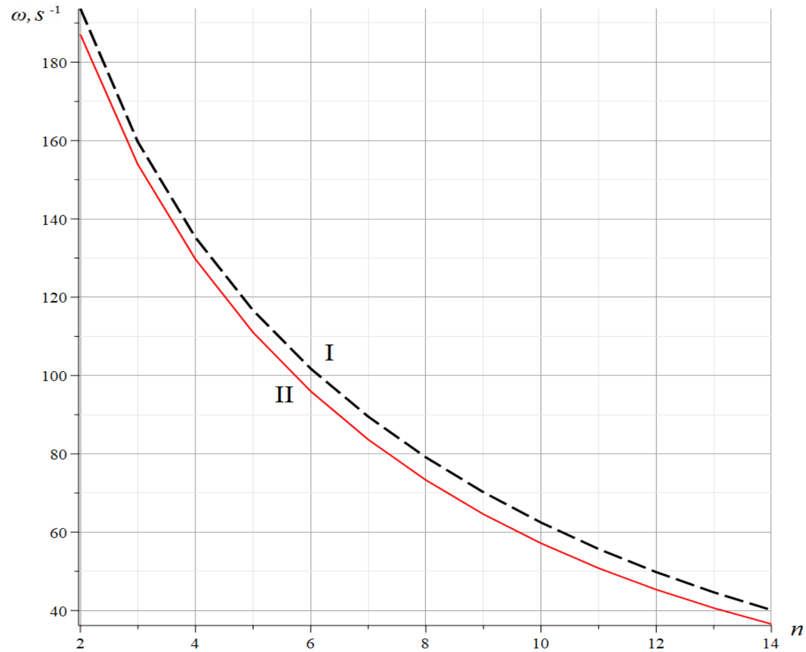

FIGURE 3. Frequency dependence on the number of panels; I is numerical solution ω_1;
II is analytical assessment ω_D, m=4.

To compare the results, we will find a relative error of $\varepsilon = (\omega_1 - \omega_D)/\omega_1$. With an increase in the number of panels in the crossbar, the error first increases and then decreases (figure 4). The nature of this change significantly depends on the number of panels in the vertical side parts of the frame. Frames with a smaller number of panels m have a maximum error at a certain value of n. In higher frames, the extremum shifts towards frames with a large span or disappears altogether. In any case, the error of the resulting formula is not large. For the selected design characteristics, this value does not exceed 11%. The greatest error of the obtained estimate is found with a small number m of frame panels in height.

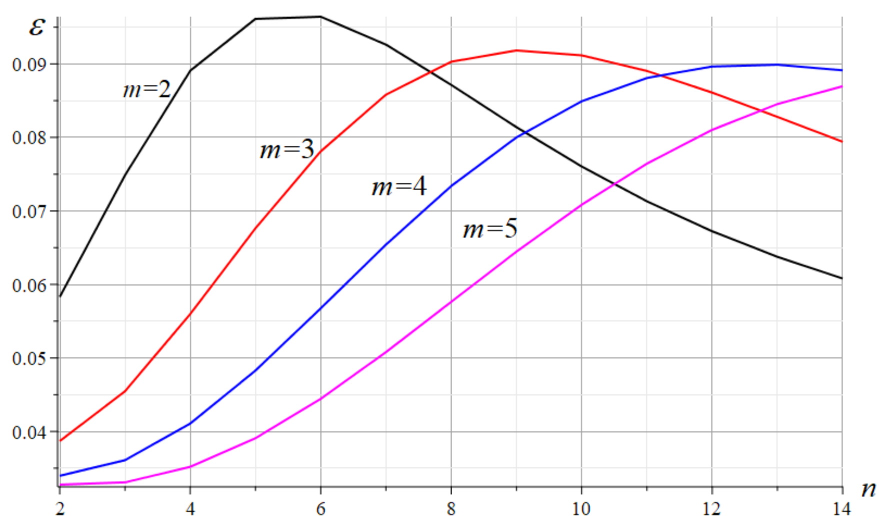

FIGURE 4. The error of Dunkerley's estimation depending on the number of panels.

CONCLUSION

The main results of the work are as follows:

1. A simple model of natural vibrations of a frame with an arbitrary number of panels has been developed.
2. Using the double induction method, the expression for the lower limit of the first vibration frequency of the structure is obtained depending on the size of the frame and the number of panels.
3. Comparison with the numerical solution showed that the found analytical solution has good accuracy, depending on the number of panels.

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