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Static Deflection of a Quadrangular Rod Pyramid: An Analytical Solution

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Abstract:

The object of the study is a spatial statically determinate truss in the form of a regular quadrangular pyramid. Along the entire perimeter of the base, the truss has vertical support posts. One corner unit is fixed on spherical support, one on cylindrical support, the others only on vertical posts. The analytical dependence of the deflection of the top of the pyramid on the number of panels at its base is derived. The load distributed over the edges and the vertical concentrated force at the vertex are considered. ***Method.*** The deflection is calculated using Mohr's integral. To determine the forces in the rods and the reactions of the supports, a system of equilibrium equations for all nodes in the projection on the coordinate axis is compiled. To generalize a series of partial solutions for trusses with a different number of panels, the induction method and operators of the Maple computer mathematics system are used. ***Results.*** A compact formula for the dependence of the deflection on the number of panels is obtained. The two coefficients of the formula have the form of polynomials in the number of panels of degree no higher than the second. The horizontal asymptote of the solution is found. Formulas are derived for the most compressed (in the edges of the structure) and the most stretched (in the base) rods.

1 Introduction

The numerical method for calculating the stress-strain state of building structures is traditional in engineering practice [1]–[6], but not the only one. In cases where a simple and adequate mathematical model can be used for an object that allows an analytical solution, the calculation by formulas has a clear advantage both in simplicity and often inaccuracy. This is because numerical methods, which are usually based on the finite element method [7]–[9], with a large number of data and the dimension of the problem, tend to inevitably accumulate rounding errors and increase the counting time. A statically determinate truss with pivot joints of the rods has a simple mathematical model. This is determined by the convenience of calculating the forces in the rods from the solution of the system of linear algebraic equations of the equilibrium of the nodes and the Mohr's integral for calculating the displacements of the nodes under the action of loads. The value of the analytical solution is related to the number of independent design parameters included in the calculation formulas. Loads, elastic characteristics of the material, and dimensions can be easily entered into the calculation formulas by replacing the numbers in the model source data with parameters and performing transformations in some system of symbolic mathematics (Maple, Mathematica [10], Reduce, etc.). For a regular system, for example, a beam truss, a significant extension of the field of applicability of the solution would be to take into account the number of panels in the solution. This would allow the calculator to combine various options to choose the optimal one in terms of rigidity, strength, stability, or material consumption. Schemes of regular statically determinate trusses are few. In 2005, Hutchinson R. G. and Fleck N. A. even announced a "hunt" for such schemes [11], [12]. For regular constructions, we apply the method of calculation induction, taking into account the order of the construction (the number of elements of periodicity). Formulas for the dependence of the deflection on the number of panels in planar trusses by induction in the Maple system are obtained in [13]–[18]. Solutions of such problems for spatial trusses [19] and the calculation of the natural frequencies of natural oscillations [20] are also known. As a rule, the method of induction determines the analytical expression for the lower bound of the first frequency by the Dockerley or Rayleigh method. This paper is the continuation of the work [19]. A scheme and calculation in the analytical form of the deflection of the spatial truss of a quadrangular pyramid are proposed (Fig. 1). Numerical calculations of the simplest pyramidal trusses of several rods are used in the design of structures of layered materials [21], [22].

2 Materials and Methods

The truss consists of four identical planar triangular lattices with vertical supports along the entire perimeter of the base of size $na \times na$, where a is the length of the base panel. The face grid contains inclined rods that are theoretically not connected at the crossing points. The truss hinges are located only on the horizontal ribs at the base and on the four inclined ribs. The height of the pyramid without taking into account the height of the support rods is nh. The total number of rods in the truss, including $4n$ vertical supports, the three rods modeling the spherical hinge of the support at corner *A*, and the cylindrical one at corner *B* is $m = 24n - 9$. The truss is statically determinate, the number of internal hinges in the structure is three times less than the number of rods and is equal to $q = 8n - 3$. At the base of the truss is $4n$ hinged nodes. The task is to derive the formula for the dependence of the deflection on the order of the structure (number n). The deflection is found using Mohr's integral under the assumption that the rods operate in a linear elastic region. To calculate the forces in the rods, use the program [23], written in the symbolic mathematics language Maple. The coordinates of the nodes and the structure of the connections of the rods are entered into the program.

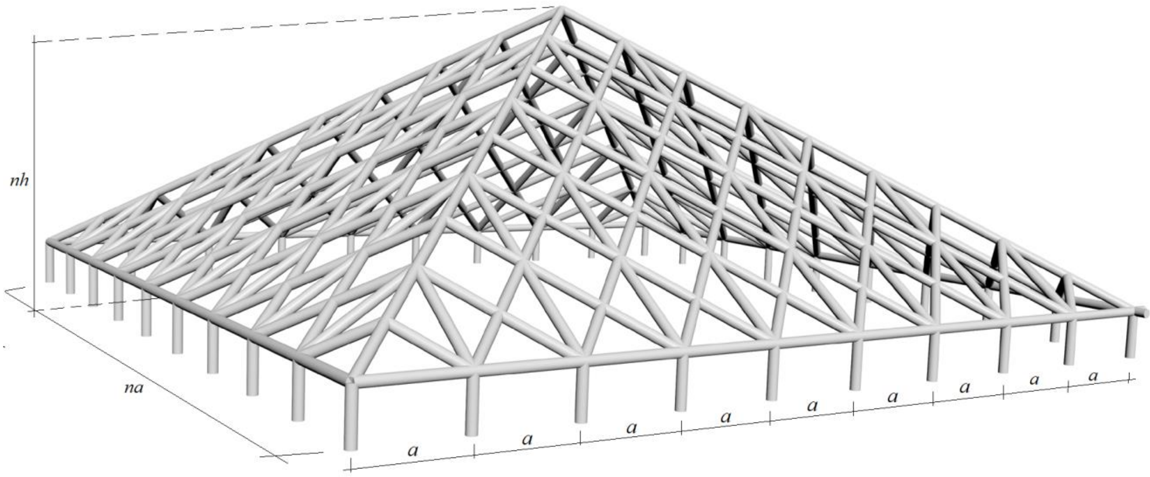

Fig. 1. Truss, *n*=9

The program uses the method of cutting out nodes to find the forces. The matrix $\mathbf{G}$ of the system of equilibrium equations of nodes is compiled in a cycle by the number of nodes

$$\mathbf{GS} = \mathbf{T}. \qquad (1)$$

Here it is indicated: $\mathbf{S}$ — the vector of all forces in the rods, including the reactions of the supports. The coefficients of the equilibrium equations of each node in the projection on three axes are assigned three rows of the matrix. The vector $\mathbf{T}$ of the right part of the system contains loads on the nodes.

3 Results and Discussion

3.1 The formula for deflection

Consider a load with an intensity P evenly distributed over the nodes of the side edges of the pyramid (Fig. 2).

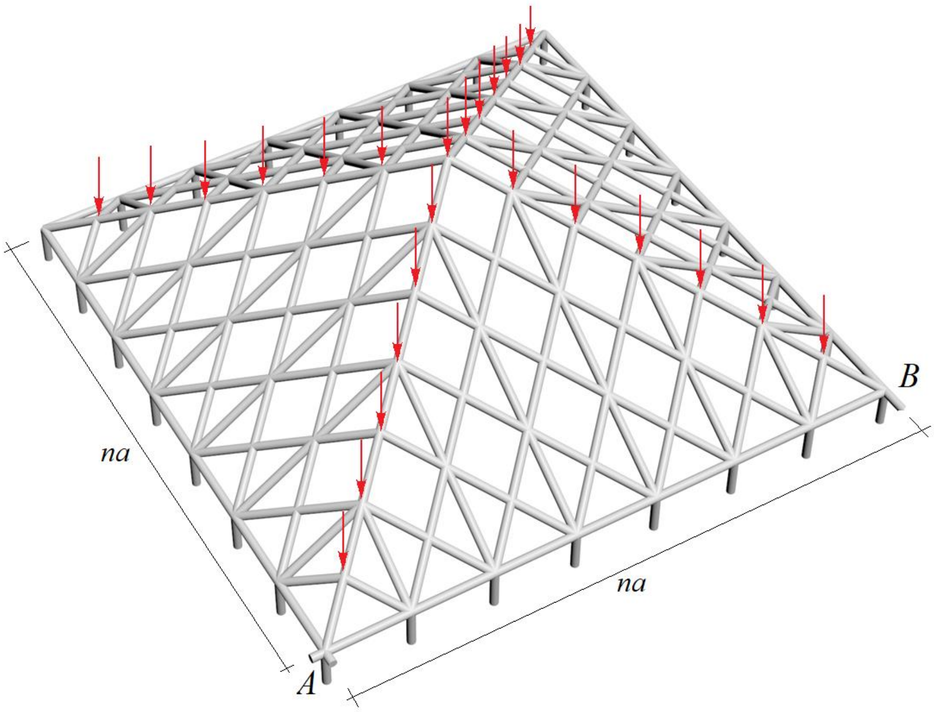

Fig. 2. Truss, *n*=7

In this case, the non-zero elements of the load vector have the form

$$T_{3i} = P,\ i = 4n+1,...,8n-3. \tag{2}$$

Lines $T_{3i-2},\ i=1,...,q$ generally contain loads on node i projected on the x-axis, and lines $T_{3i-1},\ i=1,...,q$ contain loads projected on the y-axis. In this formulation, all loads are only vertical. The solution of system (1) in the Maple program is most efficiently obtained by the inverse matrix method: $\mathbf{S} = \mathbf{G}^{-1}\mathbf{T}$. We give the distribution of forces on the rods at $a = 4$ m, $h = 1$ m. In Figure 2 stretched rods are highlighted in red, and compressed rods are highlighted in blue. The thickness of the segments is proportional to the force modules. The number shows the value of the corresponding force, rounded to one or two tenths or hundredths and assigned to the value of P. The problem is axisymmetric.

The data is given for one-quarter of the grid. The side edges of the pyramid were compressed, and the contour of the base was stretched. The method of induction can be used to obtain analytical expressions of the dependence of some forces on the number of panels.

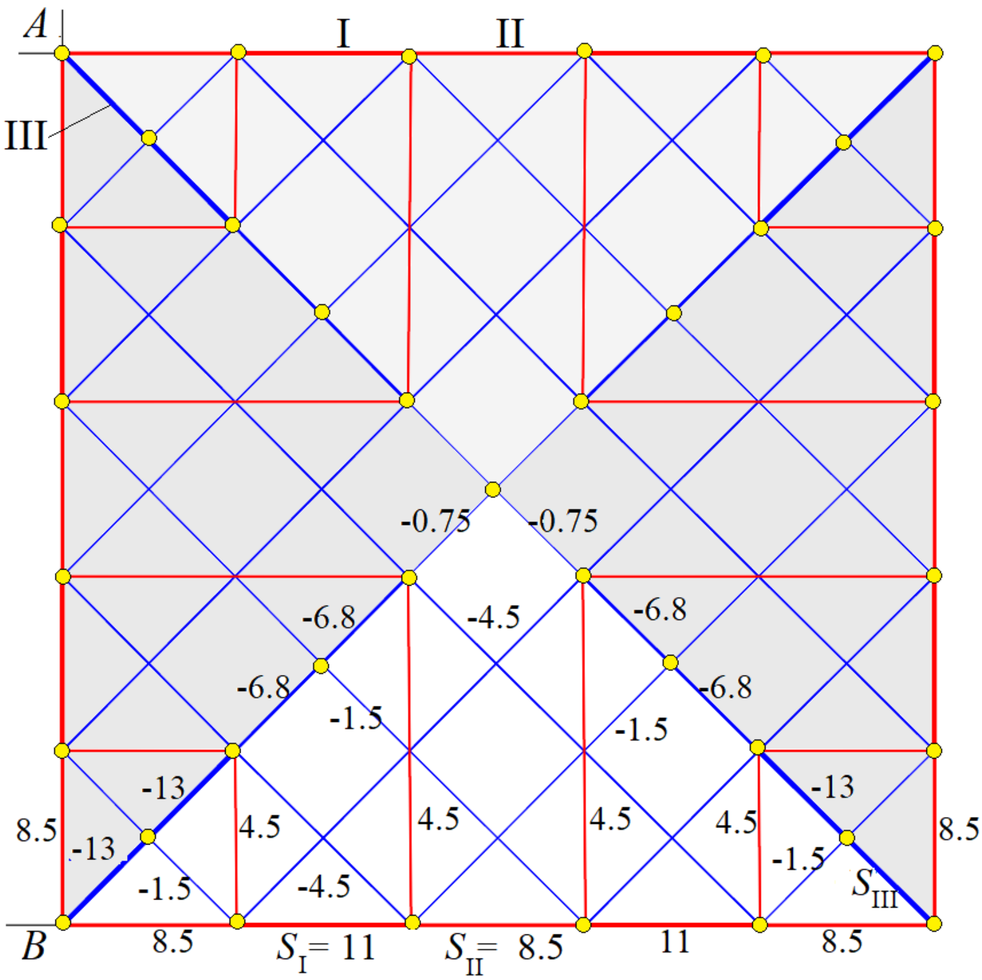

Fig. 3. Distribution of forces on the pyramid rods, *n=5*

Next, we will consider the case of an odd number of panels in the base $n = 2k + 1$. The most stretched rod in the lower contour is the middle rod II if k is an odd number, and the adjacent rod I with an even k (Fig. 3). By the inductive method, based on the results of calculating six trusses with a consistently increasing number k, we obtain the force dependences on the number k:

$$S_I = Pa(8k + 3 + 2(-1)^k)/(8h), \tag{3}$$

$$S_{II} = Pa(8k + 3 - 2(-1)^k)/(8h).$$

The most compressed rod on the side edges will always be the lower rod:

$$S_{III} = -Pc(8k + 1)/(8h), \tag{4}$$

where $c = \sqrt{2a^2 + 4h^2}$. The resulting formulas can be used to assess the stability of the structure and its strength. Equating S_{III} to the critical Euler force, we get the value of the critical load P_e per node based on the condition of local loss of stability:

$$P_e = 32\pi^2 EJh/(\mu^2 c^3 (8k + 1)), \tag{5}$$

where J is the minimum moment of inertia of the cross-section of the bars. The length reduction factor for trusses is assumed to be $\mu = 0.8$. The critical load from the loss of strength condition can be obtained from the equality of forces S_I or S_{II} to the ultimate tensile force.

When calculating the reactions of the supports, a somewhat unexpected result was found. The reactions of all vertical supports on the sides of the base, except for the corner ones, were equal to zero under uniform load. The reactions of the corner supports when the internal joints are uniformly loaded with vertical forces P are equal to $P(4n - 3)/4$. The reactions of the horizontal connections in nodes *A* and *B* are equal to zero. A similar effect is found in the triangular pyramid problem [19].

The deflection of the vertex under the action of the load is calculated using Mohr's integral

$$\Delta_k = \sum_{\alpha=1}^{m-4n-3} S_\alpha^{(P)} S_\alpha^{(1)} l_\alpha / (EF), \tag{6}$$

Here it is indicated $S_\alpha^{(1)}$ — the forces in the element number α from the action of a single vertical force applied to the top of the pyramid, $S_\alpha^{(P)}$ — force from the action of a distributed load of intensity P, EF — the longitudinal stiffness of the rods, l_α — the length of the rods. Summation is carried out for all the truss rods, except for the $4n+3$ supports ones, which are accepted as non-deformable. Calculations of trusses with a different number of panels in the base show that in all cases the desired dependence of the deflection on the number of panels has the same form

$$\Delta_k = P(C_1 a^3 + C_2 c^3) / (h^2 EF), \tag{7}$$

where the coefficients C_1 and C_2 depends only on k. The functions $C_1 = C_1(k), C_2 = C_2(k)$ are found by induction. We consistently get

$$\Delta_1 = (62a^3 + 19c^3) / (32h^2 EF),$$
$$\Delta_2 = (186a^3 + 53c^3) / (32h^2 EF),$$
$$\Delta_3 = (374a^3 + 103c^3) / (32h^2 EF),$$
$$\Delta_4 = (626a^3 + 169c^3) / (32h^2 EF),$$
$$\Delta_5 = (942a^3 + 251c^3) / (32h^2 EF),$$
$$...$$

The common terms of the coefficient sequences can be found using the operators of the Maple or Mathematica system. We get the following solution:

$$C_1 = (16k^2 + 14k + 1)/16,\ C_2 = (8k^2 + 10k + 1)/32. \tag{8}$$

Solution (7) with coefficients (8) represents the desired dependence of the deflection on the load, the size of the pyramid, and the number of panels. Similarly, the formula for the dependence of the deflection on the action of the vertical force only on the top of the pyramid is derived. As in the previous case, only in the corner supports the reactions are different from zero, and the coefficients C_1 and C_2 have the form

$$C_1 = \left(2k+1\right)/16,\ C_2 = C_1/2. \tag{9}$$

3.2 Numerical example

We plot the resulting solution for a pyramid with a fixed base side length $L = na$ and height $H = nh$ under the action of a uniform load. We introduce a dimensionless relative deflection $\Delta' = EF\Delta/(P_s L)$, where $P_s = P(4n-3)$ is the total vertical load on the structure. If $L = 80\,\mathrm{m}$ the dependence of the relative deflection on the number of panels is represented by curves in Figure 4. The horizontal asymptotes of the curves are traced. In this problem statement, the total load on the truss, the horizontal dimensions of the pyramid, and its height are fixed. With an increase in the number of panels, the load on each node decreases, and the length of the rods of the structure decreases too. The analytical form of the solution allows you to calculate the limit values of the relative deflection. The asymptotes of the constructed dependence can be detected by calculating the limit $\lim_{k\to\infty} \Delta' = (\sqrt{2}L_1^3 + 2L^3)/(32H^2 L)$, where $L_1 = \sqrt{2H^2 + L^2}$. When a concentrated force acts on the top of the pyramid, according to the solution (7), this limit is twice as large: $\lim_{k\to\infty} \Delta' = (\sqrt{2}L_1^3 + 2L^3)/(16H^2 L)$.

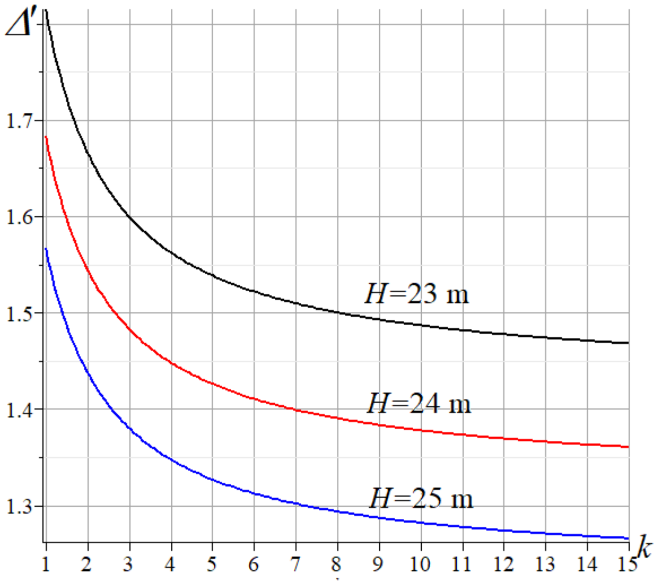

Fig. 4. Dependence of the relative deflection on the number of panels

4 Conclusions

The main results of the work are as follows.

1. A scheme of a statically definable spatial pyramid-type truss is proposed. The distribution of forces in the rods of the structure is obtained and formulas for the dependence of forces in the most stretched and compressed elements on the number of panels in the base are derived.

2. It is shown that the reactions of all the truss supports, except for the corner ones, are equal to zero.

3.The formula for the dependence of the vertical movement of the pyramid top on the number of panels for distributed and concentrated load is obtained by induction.

4. The asymptotics of the solutions are found.

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