УДК 624.074 DOI: 10.22227/1997-0935.2021.5.570-576

The analysis of dependence of the vibration frequency of a space cantilever truss on the number of panels

Mikhail N. Kirsanov, Oleg V. Vorobyev
National Research University “Moscow Power Engineering Institute” (MPEI); Moscow, Russian Federation

ABSTRACT

Introduction. The first (lowest) frequency of natural vibrations of a structure is one of its most important dynamic characteristics. Analytical solutions supplement numerical ones; they can be efficiently used to perform a rapid assessment of properties of structures, to analyze and optimize constructions and to test numerical results. A space cantilever truss consisting of three planar trusses with a rectangular grid is considered in the article. The objective is to find the analytical dependence between the frequency of natural vibrations of a structure and the number of panels. It is assumed that the truss mass is distributed among the joints. Only the vertical mass displacement is taken into account.

Materials and methods. Forces, arising in cantilever rods, are calculated by the Maple software as symbolic expressions, and the method of joint isolation is used here. The stiffness matrix is identified using the Mohr integral. Rods are assumed to be elastic, they have identical stiffness. The lower value of the vibration frequency is determined using the Dunkerley method. The final calculation formula used to identify the value of the vibration frequency is derived using the method of induction applied to a series of analytical solutions developed for trusses with a consistently increasing number of panels. When common members of sequences are found, genfunc operators of the Maple system are used. The analytical solution is compared with the numerical solution in terms of the first frequency using the analysis of the system spectrum featuring many degrees of freedom. The eigenvalues of the characteristic matrix are identified using the Eigenvalues operator from the Linear Algebra package.

Results. The comparison between the analytical values and the numerical solution shows that the Dunkerley method ensures the accuracy varying from 20 % for a small number of panels to 3 % if the number of panels exceeds ten. The size of the structure, the weight and stiffness of rods have little effect on the accuracy of the obtained values.

Conclusions. The lowest value obtained using the Dunkerley method in the form of a fairly compact formula has good accuracy, its application to a space structure with an arbitrary number of panels has a polynomial form equal to the number of panels, and it can be used in practical calculations.

KEYWORDS: cantilever truss, vibrations, frequency, Dunkerley method, Maple

Acknowledgements. This research enjoyed the support from the Interdisciplinary Scientific and Educational School of Moscow University “Fundamental and Applied Space Research”.

FOR CITATION: Kirsanov M.N., Vorobyev O.V. The analysis of dependence of the vibration frequency of a space cantilever truss on the number of panels. *Vestnik MGSU* [Monthly Journal on Construction and Architecture]. 2021; 16(5):570-576. DOI: 10.22227/1997-0935.2021.5.570-576 (rus.).

Расчет зависимости частоты колебаний пространственной консольной фермы от числа панелей

М.Н. Кирсанов, О.В. Воробьев
Национальный исследовательский университет «Московский энергетический институт» (НИУ «МЭИ»); г. Москва, Россия

АННОТАЦИЯ

Введение. Первая (низшая) частота собственных колебаний конструкции является одной из ее важнейших динамических характеристик. Аналитические решения дополняют численные, они удобны для быстрой оценки свойств сооружения и могут быть использованы для анализа и оптимизации конструкции, и выполнять роль тестов для численных результатов. Рассматривается пространственная консольная ферма, составленная из трех плоских ферм с прямоугольной решеткой. Ставится задача найти аналитическую зависимость собственной частоты колебаний конструкции от числа панелей. Предполагается, что масса фермы распределена по узлам. Учитываются только вертикальные перемещения масс.

Вестник МГСУ • ISSN 1997-0935 (Print) ISSN 2304-6600 (Online) • Том 16. Выпуск 5, 2021
Vestnik MGSU • Monthly Journal on Construction and Architecture • Volume 16. Issue 5, 2021

Материалы и методы. Расчет усилий в стержнях консоли в символьной форме производится в программе, созданной в системе компьютерной математики Maple с использованием метода вырезания узлов. Матрица жесткости находится с помощью интеграла Мора. Стержни принимаются упругими с одинаковой жесткостью. Нижняя оценка частоты колебаний определяется по методу Донкерлея. По серии аналитических решений для ферм с последовательно увеличивающимся числом панелей методом индукции выводится итоговая расчетная формула для частоты колебаний. При нахождении общих членов последовательностей используются операторы специального пакета genfunc системы Maple. Аналитическое решение сравнивается с численным решением для первой частоты, полученным из анализа спектра системы с многими степенями свободы. Собственные числа характеристической матрицы найдены с помощью оператора Eigenvalues из пакета LinearAlgebra.

Результаты. Сравнение аналитической оценки и численного решения показывает, что метод Донкерлея дает точность, меняющуюся от 20 % при малом числе панелей до 3 %, если число панелей больше десяти. Размеры конструкции, массы и жесткость стержней мало влияют на точность приведенной оценки.

Выводы. Полученная нижняя оценка по методу Донкерлея в виде достаточно компактной формулы имеет хорошую точность, ее реализация для пространственной конструкции с произвольным числом панелей в виде полиномов по числу панелей может быть применена в практических расчетах.

КЛЮЧЕВЫЕ СЛОВА: консольная ферма, колебания, частота, метод Донкерлея, Maple

Благодарности. Работа выполнена при поддержке Междисциплинарной научно-образовательной школы МГУ «Фундаментальные и прикладные космические исследования».

ДЛЯ ЦИТИРОВАНИЯ: *Кирсанов М.Н., Воробьев О.В.* The analysis of dependence of the vibration frequency of a space cantilever truss on the number of panels // Вестник МГСУ. 2021. Т. 16. Вып. 5. С. 570–576. DOI: 10.22227/1997-0935.2021.5.570-576

INTRODUCTION

The calculation of frequencies of natural vibrations of structures is usually performed in the numerical form. For regular, statically determinate trusses, having a periodic structure, some solutions of the oscillation problem are known; they are implemented in the system of computer mathematics in the form of finite formulas that encompass the number of panels in the form of a parameter [1, 2]. Problems of structures and methods used to analyze regular statically determinate rod systems were first raised in [3–5]. The reference book [6] contains various patterns of planar regular trusses and formulas needed to calculate their rigidity for an arbitrary number of panels. Separate analytical solutions, designated for planar statically determinate trusses, that entail the rigidity calculation using the induction method in the Maple system, are found in [7–9]. The induction method applied to these solutions can also be used to solve the problem of an oscillating truss [10–12]. In both cases, the matrix or the stiffness coefficient of a structure is identified using the Mohr integral. Other algorithms are also available for the analysis of regular (planar, space, and statically indeterminate) rod structures [13–15]. Practical problems of trusses, in which vibration frequencies are analyzed, are usually solved using the finite element method and associated with optimization problems [16–20]. Apart from that, one can single out the solution to problems of nonlinear oscillations [21–23].

In this paper, we consider a cantilever-type space truss having n identical panels, composed of three planar trusses (Fig. 1). The quadrature circuit of the structure's rod has the shape of an isosceles triangle (Fig. 2).

The number of truss members is $n_r = 9(n+1)$; the number of joints is $3(n+1)$. The total number of rods includes six support rods. Three members ensure the spherical support in joint A, two of them ensure the cylindrical support in joint B. Another support, a horizontally supported rod, is located in the upper belt of the structure. The truss has a statically determinate construction. The inertial properties of the structure are simulated by separate masses μ located in all truss joints, except for supports A and B. Thus, the number of the degrees of freedom of the system is $K = 3n + 1$.

MATERIALS AND METHODS

An analytical solution to the problem of the frequency spectrum of natural oscillations of systems having a large number of degrees of freedom will not be feasible because it is impossible to analytically identify the roots of algebraic equations in a degree higher than the fifth one. There are several methods that ensure an approximate solution to this problem; they can be

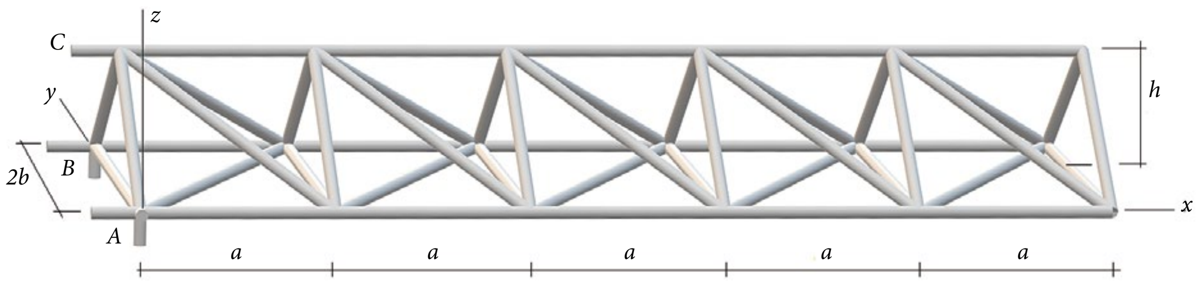

Fig. 1. Truss, n = 5

Вестник МГСУ • ISSN 1997-0935 (Print) ISSN 2304-6600 (Online) • Том 16. Выпуск 5, 2021
Vestnik MGSU • Monthly Journal on Construction and Architecture • Volume 16. Issue 5, 2021

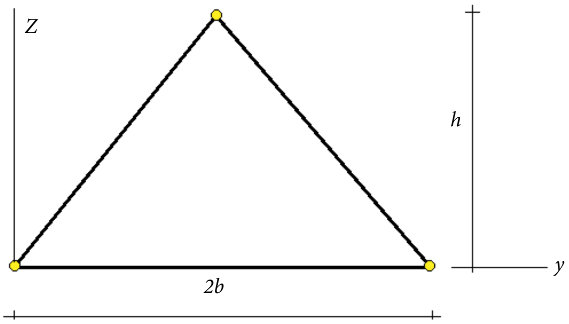

Fig. 2. The quadrature circuit of the truss

used to develop solutions in the form of finite formulas. The most suitable ones are the Dunkerley method and the Rayleigh method, which do not make it necessary to calculate all elements of the characteristic matrix. These methods require the values of diagonal elements, which can be obtained by solving the problem of vibration of individual loads applied to the joints having pre-set mass. However, the Dunkerley method offers a solution in the form of only a trace of the characteristic matrix. In the Rayleigh method, the mathematical component of the solution is somewhat more complex, although the accuracy of this method is much higher [24]. Let's consider a simpler Dunkerley method, used to identify the value of the first natural frequency of the bottom. The approximate value of Dunkerley frequency ω_D is expressed through partial frequency ω_p :

$$\omega_D = \sqrt{1/\sum_{p=1}^{K} 1/\omega_p^2}. \tag{1}$$

To calculate partial frequencies, we make a differential equation of displacement that encompasses mass μ:

$$\mu \ddot{z}_p + d_p z_p = 0,\ p = 1, ..., K,$$

where d_p is the stiffness coefficient; z_p is the mass displacement, and $\ddot{z}_p$ is the acceleration. Hence, the vibration frequency of a single load has the form: $\omega_p = \sqrt{d_p/\mu}$. The coefficient of rigidity is calculated using the Mohr integral:

$$\delta_p = 1/d_p = \sum_{j=1}^{n_r-6} \left(\tilde{S}_j^{(p)}\right)^2 l_j/(EF). \tag{2}$$

Hence, $\tilde{S}_j^{(p)}$ are the forces in the rod, where j comes from the action of a single vertical force applied to the joint where mass μ with the p number is located, E is the Young's modulus of the member material, F is the cross-sectional area of the rods. The cross sections and the material of the members (except for the three supporting ones) are assumed to be the same. It is assumed that the six supporting members will not deform. Expressions (1), (2) are used to derive the following equation:

$$\omega_D^{-2} = \mu \sum_{p=1}^{K} \delta_p = \mu\Delta. \tag{3}$$

To determine the forces included into the Mohr integral (2) in the analytical form, we use the truss calculation software [1], compiled in the language of computer mathematics Maple. The software enters the coordinates of joints, the order of connection of hinges and rods, and compiles a matrix for the system of equations of the equilibrium of joints along three axes: x, y, z. The members and hinges of the truss are numbered (Fig. 3). The origin is located in spherical support A. Coordinates are entered in cycles with a parametrically defined length:

$$x_i = x_{i+n+1} = x_{i+2n+2} = a(i-1),$$
$$y_i = 0,\ y_{i+n+1} = 2b,\ y_{i+2n+2} = b,$$
$$z_i = z_{i+n+1} = 0,\ z_{i+2n+2} = h,\ i = 1,\ ...,\ n+1.$$

The structure of the lattice is based on the numbers of the ends of corresponding members by analogy to the method of discrete mathematics used to make graphs. Longitudinal members of lower and upper belts, for example, are represented as:

$$N_i = [i,\ i+1],\ N_{i+n} = [i+n+1,\ i+n+2],$$
$$N_{i+2n} = [i+2n+2,\ i+2n+3].$$

Hence, the matrix for the system of equations of the equilibrium of joints is filled. Matrix elements are the guiding cosines of the forces arising in the members. The solution to the system generates the values of forces in the analytical form.

The analysis of trusses having a different number of panels n shows that each time value Δ represents:

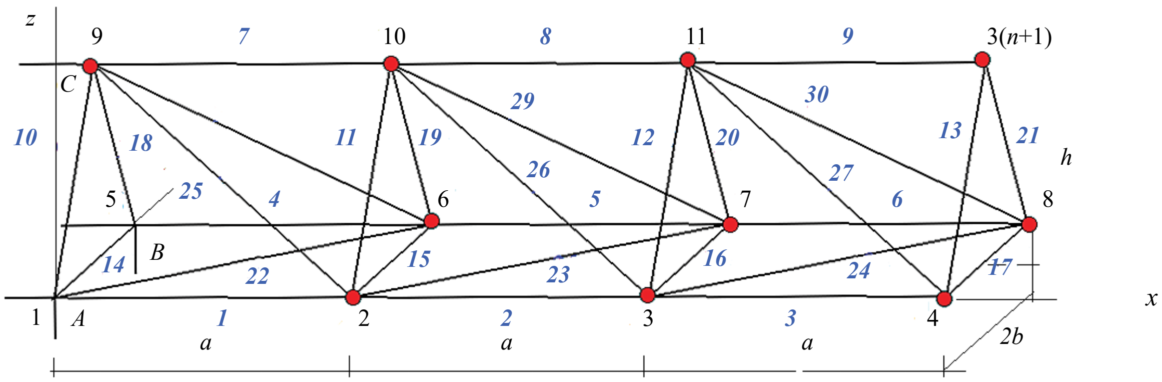

Fig. 3. Numbered joints and truss members if n = 3, and if joints and masses are highlighted

Вестник МГСУ • ISSN 1997-0935 (Print) ISSN 2304-6600 (Online) • Том 16. Выпуск 5, 2021
Vestnik MGSU • Monthly Journal on Construction and Architecture • Volume 16. Issue 5, 2021

$$\Delta = \frac{(n+1)\left(C_1a^3 + C_2b^3 + C_3c^3 + C_4d^3 + C_5f^3\right)}{EFh^2}, \qquad (4)$$

where the lengths of braces are:

$$d = \sqrt{b^2 + h^2},\ c = \sqrt{a^2 + b^2 + h^2},\ f = \sqrt{a^2 + 4b^2}.$$

For a series of analytical solutions obtained for trusses with a sequentially increasing number of panels, the coefficients in (4) are derived using the method of induction:

$$C_1 = n\left(3n^2 + n + 4\right)/8,\ C_2 = (4n+1)/2,$$
$$C_3 = 5n/4,\ C_4 = (5n+2)/4,\ C_5 = n/4.$$

As a result, we have the following dependence of the first frequency on the size of the truss, the mass, and the number of panels:

$$\omega_D = 2h\sqrt{\frac{2EF}{\mu(n+1)\left(n\left(3n^2+n+4\right)a^3 + 4(4n+1)b^3 + 10nc^3 + 2(5n+2)d^3 + 2nf^3\right)}}. \qquad (5)$$

RESEARCH RESULTS

The numerical verification can be performed in any system designated for the analysis of building structures. As a rule, such systems use the finite element method to make calculations. The truss is statically determinate, the forces in its members can be calculated using the same software in which the analytical solution (5) was obtained by converting it to the numerical mode. You can also specify all the initial geometrical, material, and inertial parameters of a structure not as symbols or integers, but as decimal numbers. The Maple system will automatically calculate everything.

The solution to the problem of vibration of a system having many degrees of freedom is related to the eigenvalue problem. Here is a differential equation describing the dynamics of a system with a finite number of degrees of freedom:

$$M_K\ddot{Z} + D_KZ = 0, \qquad (6)$$

Where Z is the displacement vector of the mass system, $\ddot{Z}$ is the acceleration vector, D_K is the stiffness matrix, and K is the number of degrees of freedom. In the case of identical masses, inertia matrix M_K is diagonal: $M_K = \mu I_k$, I_k is the unit matrix. For harmonic vibrations having frequency ω, the replacement $\ddot{Z} = -\omega^2 Z$ is valid. The matrix is the inverse of compliance matrix N, the elements of which are calculated using the Mohr integral:

$$b_{i,j} = \sum_{\alpha=1}^{n_r-6} S_\alpha^{(i)}S_\alpha^{(j)}l_\alpha/(EF),$$

where, by analogy with (2), $S_\alpha^{(i)}$ is the force in member α arising from the action of a single vertical force in joint i. Multiplying (6) by B_k, we obtain the eigenvalue problem: $B_KZ = \lambda Z$, where $\lambda = 1/(\omega^2\mu)$ are the eigenvalues of the matrix. Eigenvalues, a special operator, is used to determine the eigenvalue of a matrix in the Maple system.

DISCUSSION AND CONCLUSIONS

We present the comparison of numerical and analytical methods. Consider a steel truss having mass μ = 1000 kg in the joint. We assume the modulus of elasticity $E = 2 \cdot 10^5$ MPa, $F = 10.9$ sm^2, $a = 2$ m, $h = b = 1$ m. In the graph (Fig. 4), the Dunkerley curve ω_D, constructed according to formula (5), is located below the curve of the first frequency ω_1, obtained numerically

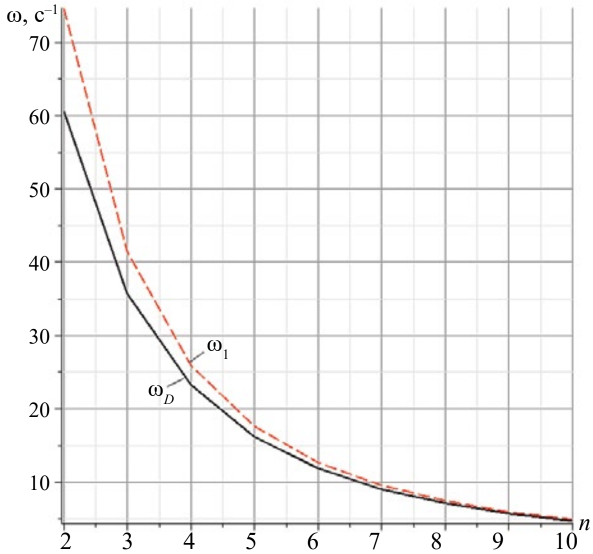

Fig. 4. Comparison with the frequency dependence on the number of panels obtained numerically

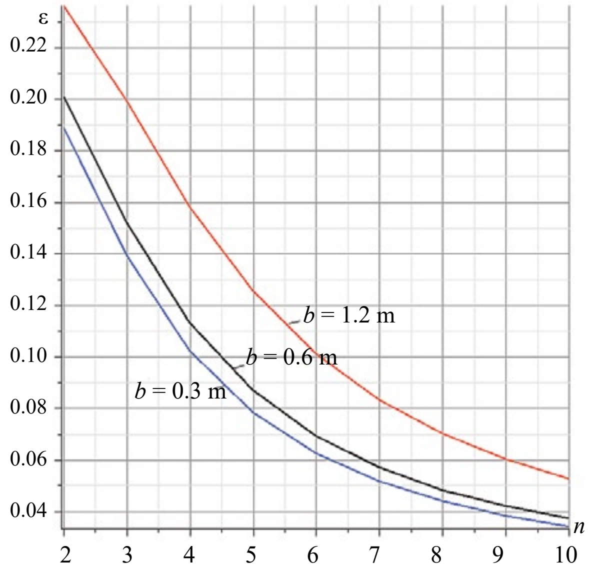

Fig. 5. Dependence of accuracy on the number of panels

Вестник МГСУ • ISSN 1997-0935 (Print) ISSN 2304-6600 (Online) • Том 16. Выпуск 5, 2021
Vestnik MGSU • Monthly Journal on Construction and Architecture • Volume 16. Issue 5, 2021

merically. As the number of panels increases, these two solutions converge.

To estimate the error value, we enter value

$$\varepsilon = (\omega_1 - \omega_D)/\omega_1 .$$

Fig. 5 shows the dependence of the error on the n number for three values of the cantilever width. The accuracy increases quite rapidly along with the number of panels, approaching a quite acceptable error of a few percent. Thus, the value of the obtained analytical solution increases precisely in those cases when numerical solutions have the least accuracy due to the increase in the dimensions of the matrices used (the matrices of the system of equilibrium equations and the stiffness matrix) with the inevitable accumulation of rounding errors. Increasing the width of truss b does not significantly reduce the accuracy of the lower value. The same pattern applies to dependence of accuracy on height h.

The spatial truss model having masses concentrated at the joints and a restriction imposed on the vertical displacement of masses is probably the simplest one, if we do not consider planar truss models. Another simplification of the accepted model, the equality of the stiffness of all members, is not essential. The obtained solution can be generalized by applying relative stiffness coefficients to individual groups of members having the same length. The formula (4) will be as follows:

$$\Delta = (n+1)\left(C_1 a^3/\gamma_a + C_2 b^3/\gamma_b + C_3 c^3/\gamma_c + C_4 d^3/\gamma_d + C_5 f^3/\gamma_f\right)/\left(EF_0 h^2\right),$$

where the stiffness of rods is expressed in terms of reduced stiffness: $EF_a = \gamma_a EF_0$, ..., $EF_a = \gamma_a EF_0$.

The above algorithm used to derive the formula applied to identify the dependence of the natural vibration frequency on the number of panels shows that the result has a compact form and a sufficiently high accuracy. An attempt to employ the Rayleigh method in a similar formulation using Maple transformations in the symbolic form shows that the final formula turns out to be cumbersome and not very convenient for use in practice.

REFERENCES

1. Tinkov D.V. *Analytical solutions of problems on natural frequencies of oscillations of regular rod systems* : dissertation abstract candidate of technical sciences. Moscow, National Research University MPEI, 2019; 20. (rus.).

2. Vorobyev O. Bilateral analytical estimation of first frequency of a plane truss. *Construction of Unique Buildings and Structures.* 2020; 92:9204. DOI: 10.18720/CUBS.92.4

3. Hutchinson R.G., Fleck N.A. Microarchitectured cellular solids — the hunt for statically determinate periodic trusses. *ZAMM*. 2005; 85(9):607-617. DOI: 10.1002/zamm.200410208

4. Hutchinson R.G., Fleck N.A. The structural performance of the periodic truss. *Journal of the Mechanics and Physics of Solids*. 2006; 54(4):756-782. DOI: 10.1016/j.jmps.2005.10.008

5. Zok F.W., Latture R.M., Begley M.R. Periodic truss structures. *Journal of the Mechanics and Physics of Solids*. 2016; 96:184-203. DOI: 10.1016/j.jmps.2016.07.007

6. Kirsanov M. *Trussed Frames and Arches: Schemes and Formulas.* UK, Cambridge Scholars Publishing, 2020; 186.

7. Voropai R.A., Kazmiruk I.Yu. Analytical study of the horizontal stiffness of the flat statically determinate arch truss. *Bulletin of Scientific Conferences.* 2016; 2-1(6):10-12.

8. Voropai R.A. Analysis of the deflection of the regular truss with cross type lattice. *Science Almanac*. 2016; 4-3(18):238-240. DOI: 10.17117/na.2016.04.03.238 (rus.).

9. Shipaeva A.S. Calculation of the deflection of girder beam loaded on the bottom flange in the system Maple. *Science Almanac.* 2016; 5-3(19):236-239. DOI: 10.17117/na.2016.05.03.236 (rus.).

10. Kanatova M.I. Frequency equation and analysis of vibrations of a flat girder. *Trends in Applied Mechanics and Mechatronics*. 2015; 1:31-34. (rus.).

11. Kirsanov M.N., Tinkov D.V. Analysis of the natural frequencies of oscillations of a planar truss with an arbitrary number of panels. *Vestnik MGSU* [Monthly Journal on Construction and Architecture]. 2019; 14(3):(126):284-292. DOI: 10.22227/1997-0935.2019.3.284-292

12. Petrichenko E.A. The lower limit of the frequency of natural oscillations of the Fink truss. *Structural mechanics and structures*. 2020; 3(26):21-29.

13. Rybakov L.S., Mishustin I.V. Natural vibrations of plane regular elastic trusses of orthogonal structure. *Mechanics of composite materials and structures.* 1999; 5(2):3-16.

14. Rybakov L.S., Mishustin I.V. Application of the concentrated mass method to the analysis of natural elastic vibrations of one regular truss structure. *Mechanics of composite materials and structures.* 1999; 5(4):51-64.

15. Mishustin I.V., Rybakov L.S. Oscillations of plane elastic trusses of orthogonal structure. *Rigid body mechanics.* 2003; 2:168-184.

16. Kilikevičius A., Bačinskas D., Jurevičius M., Kilikevičiene K., Fursenko A., Jakaitis J. et al. Field testing and dynamic analysis of old continuous truss steel bridge. *Baltic Journal of Road and Bridge Engineering.* 2018; 13(1):54-66. DOI: 10.3846/bjrbe.2018.394

Вестник МГСУ • ISSN 1997-0935 (Print) ISSN 2304-6600 (Online) • Том 16. Выпуск 5, 2021
Vestnik MGSU • Monthly Journal on Construction and Architecture • Volume 16. Issue 5, 2021

17. Bačinskas D., Turla V., Kilikevičius A., Ragauskas P., Jurevièius M. Dynamic testing of railway truss-bridge. *Journal of Vibroengineering*. 2014; 16(6):2649-2657.

18. Vaez S.R.H., Mehanpour H., Fathali M.A. Reliability assessment of truss structures with natural frequency constraints using metaheuristic algorithms. *Journal of Building Engineering.* 2020; 28:101065. DOI: 10.1016/j.jobe.2019.101065

19. Lieu Q.X., Do D.T.T., Lee J. An adaptive hybrid evolutionary firefly algorithm for shape and size optimization of truss structures with frequency constraints. *Computers & Structures.* 2018; 195:99-112. DOI: 10.1016/j.compstruc.2017.06.016

20. Pham H.A. Truss optimization with frequency constraints using enhanced differential evolution based on adaptive directional mutation and nearest neighbor comparison. *Advances in Engineering Software.* 2016; 102:142-154. DOI: 10.1016/j.advengsoft.2016.10.004

21. Ufimtsev E., Voronina M. Research of total mechanical energy of steel roof truss during structurally nonlinear oscillations. *Procedia Engineering.* 2016; 150:1891-1897. DOI: 10.1016/j.proeng.2016.07.188

22. Chen J., Zhang W., Zhang Y.F. Equivalent continuum model and nonlinear breathing vibrations of rotating circular truss antenna subjected to thermal excitation. *Thin-Walled Structures.* 2020; 157:107127. DOI: 10.1016/j.tws.2020.107127

23. Liu M., Cao D., Zhang X., Wei J., Zhu D. Nonlinear dynamic responses of beamlike truss based on the equivalent nonlinear beam model. *International Journal of Mechanical Sciences.* 2021; 194:106197. DOI: 10.1016/j.ijmecsci.2020.106197

24. Bachmann H. *Vibration Problems in Structures: Practical Guidelines*. Birkhäuser Verlag, Basel, 1995; 234.

Received February 18, 2021.
Adopted in revised form on April 29, 2021.
Approved for publication on April 29, 2021

Bionotes: **Mikhail N. Kirsanov** — Doctor of Physical and Mathematical Sciences, Professor of the Department of Robotics, Mechatronics, Dynamics and Strength of Machines; **National Research University "Moscow Power Engineering Institute" (MPEI)**; 14 Krasnokazarmennaya st., Moscow, 111250, Russian Federation; SPIN-code: 8679-6853, Scopus: 16412815600, ResearcherID: H-9967-2013, ORCID: 0000-0002-8588-3871, Google Scholar: FfoNG-FwAAAAJ, IstinaResearcherID: 2939132; C216@ya.ru;

Oleg V. Vorobyev — postgraduate student of the Department of Robotics, Mechatronics, Dynamics and Strength of Machines; **National Research University "Moscow Power Engineering Institute" (MPEI)**; 14 Krasnokazarmennaya st., Moscow, 111250, Russian Federation; ID RISC: 1091660, ORCID: 0000-0002-5220-1264; olvarg@mail.ru.

ЛИТЕРАТУРА

1. *Тиньков Д.В.* Аналитические решения задач о собственных частотах колебаний регулярных стержневых систем : автореф. дис. … канд. техн. наук. М. : НИУ МЭИ, 2019. 20 с.

2. *Vorobev O.* Bilateral analytical estimation of first frequency of a plane truss // Construction of Unique Buildings and Structures. 2020. Vol. 92. Article № 9204. DOI: 10.18720/CUBS.92.4

3. *Hutchinson R.G., Fleck N.A.* Microarchitectured cellular solids — the hunt for statically determinate periodic trusses // ZAMM. 2005. Vol. 85. Issue 9. Pp. 607–617. DOI: 10.1002/zamm.200410208

4. *Hutchinson R.G., Fleck N.A.* The structural performance of the periodic truss // Journal of the Mechanics and Physics of Solids. 2006. Vol. 54. Issue 4. Pp. 756–782. DOI: 10.1016/j.jmps.2005.10.008

5. *Zok F.W., Latture R.M., Begley M.R.* Periodic truss structures // Journal of the Mechanics and Physics of Solids. 2016. Vol. 96. Pp. 184–203. DOI: 10.1016/j.jmps.2016.07.007

6. *Kirsanov M.* Trussed frames and arches: Schemes and formulas. UK : Cambridge Scholars Publishing, 2020. 186 p.

7. *Voropai R.A., Kazmiruk I.Yu.* Analytical study of the horizontal stiffness of the flat statically determinate arch truss // Bulletin of Scientific Conferences. 2016. № 2–1 (6). Pp. 10–12.

8. *Воропай Р.А.* Анализ прогиба регулярной фермы с крестообразной решеткой // Научный альманах. 2016. № 4–3 (18). С. 238–240. DOI: 10.17117/na.2016.04.03.238

9. *Шипаева А.С.* Расчет прогиба балочной фермы, загруженной по нижнему поясу в системе Maple // Научный альманах. 2016. № 5–3 (19). С. 236–239. DOI: 10.17117/na.2016.05.03.236

10. Канатова *М.И.* Частотное уравнение и анализ колебаний плоской балочной фермы // Trends in Applied Mechanics and Mechatronics. 2015. С. 31–34.

11. *Kirsanov M.N., Tinkov D.V.* Analysis of the natural frequencies of oscillations of a planar truss with an arbitrary number of panels // Вестник МГСУ. 2019. Т. 14. № 3 (126). С. 284–292. DOI: 10.22227/1997-0935.2019.3.284-292

12. *Петриченко Е.А.* Нижняя граница частоты собственных колебаний фермы Финка // Строительная механика и конструкции. 2020. № 3 (26). С. 21–29.

13. *Рыбаков Л.С., Мишустин И.В.* Собственные колебания плоских регулярных упругих ферм ортогональной структуры // Механика композиционных материалов и конструкций. 1999. Т. 5. № 2. С. 3–16.

14. *Рыбаков Л.С., Мишустин И.В.* Применение метода сосредоточенных масс к анализу собственных упругих колебаний одной регулярной ферменной структуры // Механика композиционных материалов и конструкций. 1999. Т. 5. № 4. С. 51–64.

15. *Мишустин И.В., Рыбаков Л.С.* Колебания плоских упругих ферм ортогональной структуры // Известия Российской академии наук. Механика твердого тела. 2003. № 2. С. 168–184.

16. *Kilikevičius A., Bačinskas D., Jurevičius M., Kilikevičiene K., Fursenko A., Jakaitis J. et al.* Field testing and dynamic analysis of old continuous truss steel bridge // Baltic Journal of Road and Bridge Engineering. 2018. Vol. 13. Issue 1. Pp. 54–66. DOI: 10.3846/bjrbe.2018.394

17. *Bačinskas D., Turla V., Kilikevičius A., Ragauskas P., Jurevièius M.* Dynamic testing of railway truss-bridge // Journal of Vibroengineering. 2014. Vol. 16. Issue 6. Pp. 2649–2657.

18. *Vaez S.R.H., Mehanpour H., Fathali M.A.* Reliability assessment of truss structures with natural frequency constraints using metaheuristic algorithms // Journal of Building Engineering. 2020. Vol. 28. P. 101065. DOI: 10.1016/j.jobe.2019.101065

19. *Lieu Q.X., Do D.T T., Lee J.* An adaptive hybrid evolutionary firefly algorithm for shape and size optimization of truss structures with frequency constraints // Computers & Structures. 2018. Vol. 195. Pp. 99–112. DOI: 10.1016/j.compstruc.2017.06.016

20. *Pham H.A.* Truss optimization with frequency constraints using enhanced differential evolution based on adaptive directional mutation and nearest neighbor comparison // Advances in Engineering Software. 2016. Vol. 102. Pp. 142–154. DOI: 10.1016/j.advengsoft.2016.10.004

21. *Ufimtsev E., Voronina M.* Research of total mechanical energy of steel roof truss during structurally nonlinear oscillations // Procedia Engineering. 2016. Vol. 150. Pp. 1891–1897. DOI: 10.1016/j.proeng.2016.07.188

22. *Chen J., Zhang W., Zhang Y.F.* Equivalent continuum model and nonlinear breathing vibrations of rotating circular truss antenna subjected to thermal excitation // Thin-Walled Structures. 2020. Vol. 157. P. 107127. DOI: 10.1016/j.tws.2020.107127

23. *Liu M., Cao D., Zhang X., Wei J., Zhu D.* Nonlinear dynamic responses of beamlike truss based on the equivalent nonlinear beam model // International Journal of Mechanical Sciences. 2021. Vol. 194. P. 106197. DOI: 10.1016/j.ijmecsci.2020.106197

24. *Bachmann H.* Vibration problems in structures: Practical guidelines. Birkhäuser Verlag, Basel, 1995. 234 p.

Поступила в редакцию 18 февраля 2021 г.
Принята в доработанном виде 29 апреля 2021 г.
Одобрена для публикации 29 апреля 2021 г.

Об авторах: **Михаил Николаевич Кирсанов** — доктор физико-математических наук, профессор кафедры робототехники, мехатроники, динамики и прочности машин; **Национальный исследовательский университет «Московский энергетический институт» (НИУ «МЭИ»)**; 111250, г. Москва, ул. Красноказарменная, д. 14; SPIN-код: 8679-6853, Scopus: 16412815600, ResearcherID: H-9967-2013, ORCID: 0000-0002-8588-3871, Google Scholar: FfoNGFwAAAAJ, IstinaResearcherID: 2939132; C216@ya.ru;

Олег Владимирович Воробьев — аспирант кафедры робототехники, мехатроники, динамики и прочности машин; **Национальный исследовательский университет «Московский энергетический институт» (НИУ «МЭИ»)**; 111250, г. Москва, ул. Красноказарменная, д. 14; РИНЦ ID: 1091660, ORCID: 0000-0002-5220-1264; olvarg@mail.ru.