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## *International Journal for Computational Civil and Structural Engineering*

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## СОДЕРЖАНИЕ

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# NUMERICAL SOLUTION OF THE PROBLEM OF BEAM ANALYSIS WITH THE USE OF B-SPLINE FINITE ELEMENT METHOD

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**Abstract**: Numerical solution of the problem of beam analysis (bending analysis of the Bernoulli beam) with the use of B-spline finite element method is under consideration in the distinctive paper. The original continual and finite element formulations of the problem are given, some actual aspects of construction of normalized basis functions of a B-spline are considered, the corresponding local constructions for an arbitrary finite element are described, some information about the numerical implementation and an example of analysis are presented.

**Keywords**: wavelet-based finite element method, B-spline finite element method, finite element method, B-spline, numerical solution, beam analysis

# ЧИСЛЕННОЕ РЕШЕНИЕ ЗАДАЧИ О ПОПЕРЕЧНОМ ИЗГИБЕ БАЛКИ НА ОСНОВЕ ВЕЙВЛЕТ-РЕАЛИЗАЦИИ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ С ИСПОЛЬЗОВАНИЕМ B-СПЛАЙНОВ

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**Аннотация**: В настоящей статье рассматривается численное решение задачи о поперечном изгибе балки Бернулли на основе вейвлет-реализации метода конечных элементов с использованием B-сплайнов. Приведены исходные континуальная и конечноэлементная постановки задачи, рассмотрены некоторые актуальные вопросы построения нормализованных базисных функций B-сплайна, описаны соответствующие локальные построения для произвольного конечного элемента, представлены некоторые сведения о численной реализации и пример расчета.

**Ключевые слова**: вейвлет-реализация метода конечных элементов, метод конечных элементов, B-сплайны, численное решение, изгиб балки

## INTRODUCTION

As is known, the B-spline in a given simple knot sequence can be constructed by employing piecewise polynomials between the knots and joining them together at the knots [1].

Compared with commonly used Daubechies wavelets [2-6] B-spline wavelet on interval (BSWI) has explicit expressions, facilitating the calculation of coefficient integration and differentiation [1]. Besides, the multiresolution and localization properties of BSWI can also supply some superiority for engineering structural analysis [1]. The early applications of spline can be found, for instance, in papers of H. Antes [7], J.G. Han [8, 9, 25], Y. Huang [8, 9], W.X. Ren [8, 9]. The spline wavelet finite element method was further developed in papers of D.P. Chen [26], X.F. Chen [10, 11, 13-16, 21, 22, 24], H.B. Dong [21], J.G. Han [23], Y.M. He [15], Z.H. He [16], Z.J. He [10, 11, 13-15, 21, 22, 24], Y. Huang [23, 25], Z.S. Jiang [20], B. Li [11, 13, 15, 21], M. Liang [17, 19], J.Q. Long [18], G. Ma [18], T. Matsumoto [18, 20], S.T. Mau [28], H.H. Miao [13], Q,M. Mo [16], T.H.H. Pian [26-28], K.Y. Qi [21], W.X. Ren [23, 25], K. Sumihara [27], P. Tong [28], Y.W. Wang [20], J.W. Xiang [10-12, 15-20], Z.B. Yang [13, 14, 22], X.W. Zhang [14, 22, 24], Y.H. Zhang [10], Y.T. Zhong [12].

The distinctive paper is devoted to numerical solution of the problem of beam analysis (bending analysis of the Bernoulli beam) with the use of B-spline finite element method.

## 1. FORMULATIONS OF THE PROBLEM

The unknown function of the beam deflections $y(x)$, caused by the load $q(x)$, can be defined using the condition for the minimum energy functional of the beam $\Phi(y)$ (i.e. unknown function provides a minimum value for this functional):

$$\Phi(y)=\frac{1}{2}\int_{0}^{l}[EJ(y'')^2+\beta y^2]dx-\int_{0}^{l}q(x)y dx, \quad (1.1)$$

where $EJ(x)$ is the bending stiffness of the beam; $\beta$ is the coefficient of elasticity of the base (coefficient of bedding); $q(x)$ is the given load; $l$ is the length of the beam; $x$ is coordinate along the length of the beam.

Let us divide the interval [o, $l$], occupied by the beam into $N_e$ parts (elements); $h_e = l/N_e$ is the length of the element. Let us also divide each element into $N_k$ parts, for example, $N_k = 5$ (Figure 1). Let us introduce the following notation system: $i_e$ is the element number; $x_1(i_e)$ is the coordinate of the starting point; $x_6(i_e)$ is the coordinate of the end point of the element number $i_e$, respectively. We take $y_i$ and $y'_i$ as unknowns at boundary points $i = 1,6$. We take $y_i$, $i = 2, 3, 4, 5$ as unknowns at the inner points. Thus, the number of unknowns per element with such discretization is defined by formula

$$N = N_k - 1 + 2 \cdot 2 = N_k + 3 = 8.$$

The number of boundary points for all elements is equal to

$$N_b = N_e + 1.$$

The number of interior points for all elements is equal to

$$N_p = N_e (N_k - 1).$$

The total (global) number of unknowns with such a discretization turns out to be equal to

$$N_g = N_p + 2N_b.$$

Thus, we have

$$\Phi(y) = \sum_{i_e=1}^{N_e} \Phi_{i_e}(y),$$

$$\Phi_{i_e}(y)=\frac{1}{2}\int_{x_1(i_e)}^{x_5(i_e)}[EJ(y'')^2+\beta y^2]\,dx-\int_{x_1(i_e)}^{x_5(i_e)}qy\,dx; \quad (1.2)$$

## 2. SOME ASPECTS OF THE CONSTRUCTION OF NORMALIZED BASIS FUNCTIONS OF THE B-SPLINE

The construction of B-spline basic functions is determined by the recursive Cox-de Boer formulas:

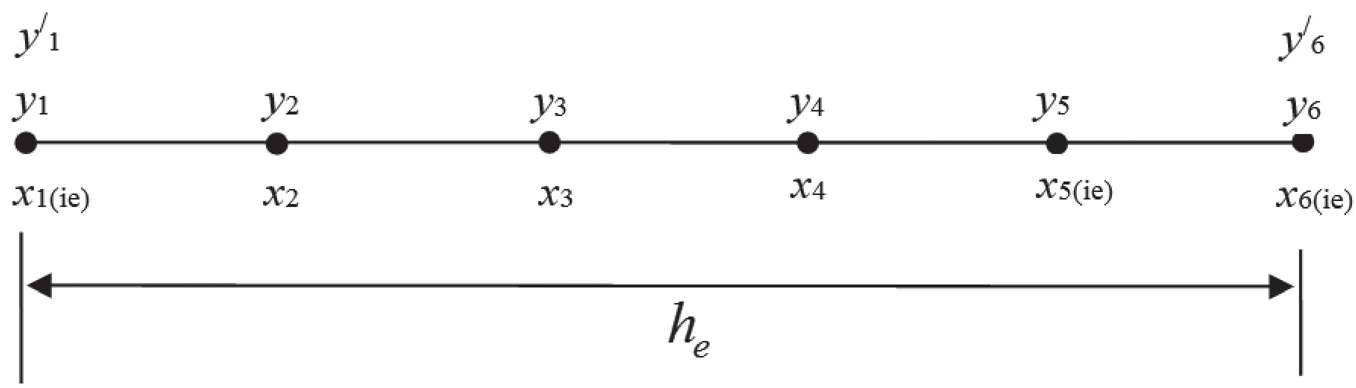



*Figure 1. Finite element discretication (sample).*

$$k=1: \quad \varphi_{i,1}(t)=\begin{cases}1, & x_i \le t < x_{i+1}\\ 0, & t<x_i \vee t \ge x_{i+1}\end{cases}, \tag{2.1}$$

$$k \ge 2: \quad \varphi_{i,k}(t)=\frac{(t-x_i)\varphi_{i,k-1}(t)}{x_{i+k-1}-x_i}+ \frac{(x_{i+k}-t)\varphi_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}}. \tag{2.2}$$

We will consider such a construction for the case $x_i = i$ are integers. Let us note that,

$$\varphi_{i,k}(t)=\varphi_{0,k}(t-i)$$

and therefore, recursive formulas (2.1)-(2.2) can be represented in the form

$$k=1: \quad \varphi_{0,1}(t)=\begin{cases}1, & 0 \le t<1\\ 0, & t<0 \vee t \ge 1\end{cases}, \tag{2.3}$$

$$k \ge 2: \quad \varphi_{0,k}(t)=\frac{1}{k-1}[t\cdot\varphi_{0,k-1}(t)+ (k-t)\varphi_{0,k-1}(t-1)]. \tag{2.4}$$

The function $\varphi_{0,1}(t)$ can be represented by formula

$$\varphi_{0,1}(t)=\frac{1}{2}[\operatorname{sign}(t)-\operatorname{sign}(t-1)]. \tag{2.5}$$

Let us denote by $\Delta_1$ the operator of the first difference. Then we have

$$\varphi_{0,1}(t)=-\frac{1}{2}\Delta_1 \operatorname{sign}(t). \tag{2.6}$$

We can substitute formula (2.5) into (2.4) in order to determine $\varphi_{0,2}(t)$:

$$\varphi_{0,2}(t)=1\cdot[t\cdot\varphi_{0,1}(t)+(2-t)\varphi_{0,1}(t-1)]=$$
$$=\frac{1}{2}\{t\cdot[\operatorname{sign}(t)-\operatorname{sign}(t-1)]+(2-t)[\operatorname{sign}(t-1)-\operatorname{sign}(t-2)]\}=$$
$$=\frac{1}{2}[t\operatorname{sign}(t)-2(t-1)\operatorname{sign}(t-1)+$$
$$(t-2)\operatorname{sign}(t-2)]=\frac{1}{2}[|t|-2|t-1|+|t-2|].$$

Let us denote by $\Delta_2$ the operator of the second difference. Then we have

$$\varphi_{0,2}(t)=\frac{1}{2}[|t|-2|t-1|+|t-2|=\frac{1}{2}\Delta_2|t-1|. \tag{2.7}$$

We can define function $\varphi_{0,3}(t)$:

$$\varphi_{0,3}(t)=\frac{1}{2}[t\cdot\varphi_{0,2}(t)+(3-t)\varphi_{0,2}(t-1)].$$

Omitting intermediate calculations, we get

$$\varphi_{0,3}(t)=\frac{1}{4}[t\cdot|t|-3(t-1)|t-1|+ +3(t-2)|t-2|-(t-3)|t-3|]=$$
$$=-\frac{1}{2!}\frac{1}{2}\Delta_1\Delta_2((t-1)|t-1|). \tag{2.8}$$

Based on formulas (2.8) and (2.4), we can define the function

$$\varphi_{0,4}(t)=\frac{1}{3}[t\cdot\varphi_{0,3}(t)+(4-t)\varphi_{0,3}(t-1)].$$

Omitting intermediate calculations, we get

$$\varphi_{0,4}(t)=$$
$$=\frac{1}{2\cdot 3}\cdot\frac{1}{2}[t^2\cdot|t|-4(t-1)^2|t-1|+ +6(t-2)^2|t-2|-4(t-3)^2|t-3|+ +(t-4)^2|t-4|]=$$
$$=\frac{1}{3!}\frac{1}{2}(\Delta_2)^2((t-2)^2|t-2|). \tag{2.9}$$

It can be proved that for even $k = 2m$ we have

$$\varphi_{0,k}(t)=\frac{1}{(2m-1)!}\frac{1}{2}(\Delta_2)^m((t-m)^{2m-2}|t-m|) \tag{2.10}$$

and for odd (uneven) $k = 2m + 1$ we have

$$\varphi_{0,k}(t)=-\frac{1}{(2m)!}\frac{1}{2}\Delta_1(\Delta_2)^m((t-m)^{2m-1}|t-1|). \tag{2.11}$$

Note that $\varphi_{0,k}(t)$ is a polynomial of degree $k – 1$ with bounded support and, as follows from the difference operator, this support is equal to the interval $[0, k]$. In addition, we should note the following property of B-spline basis functions:

$$\sum_i \varphi_{0,k}(t-i)\equiv 1 \text{ for arbitrary } t. \tag{2.12}$$

## 3. LOCAL CONSTRUCTIONS FOR ARBITRARY FINITE ELEMENT

Let us introduce local coordinates:

$$t=\frac{x-x_{1(i_e)}}{h_e}, \; x_{1(i_e)} \le x \le x_{6(i_e)}, \; 0 \le t \le 1 .$$

In this case, we have the following relations:

$$\begin{cases} x = x_{1(i_e)} \Rightarrow t = 0 \\ x = x_2 \Rightarrow t = 0.2 \\ x = x_3 \Rightarrow t = 0.4 \\ x = x_4 \Rightarrow t = 0.6 \\ x = x_5 \Rightarrow t = 0.8 \\ x = x_{6(i_e)} \Rightarrow t = 1 \end{cases},$$

$$\begin{matrix} \frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = \frac{1}{h_e} \frac{d}{dt} \\ \frac{d^p}{dx^p} = \frac{1}{h_e^p} \frac{d^p}{dt^p} \end{matrix}, \quad dx = h_e \cdot dt. \qquad (3.1)$$

Since the number of unknowns on the element is equal to $N = 8$, we use a B-spline of the seventh degree in order to represent the unknown deflection function. Let us use the following notation:

$$\varphi(t) = \varphi_{0,8}(t+4);$$

$$\begin{aligned} \varphi(t) = {} & \frac{1}{7!} \frac{1}{2} (\Delta_2)^4 (t^6 \mid t \mid) = \\ & = \frac{1}{2 \cdot 7!} [(t+4)^6 \mid t+4 \mid - \\ & - 8(t+3)^6 \mid t+3 \mid + \\ & + 28(t+2)^6 \mid t+2 \mid - \\ & - 56(t+1)^6 \mid t+1 \mid + 70 t^6 \mid t \mid - \\ & - 56(t-1)^6 \mid t-1 \mid + 28(t-2)^6 \mid t-2 \mid - \\ & - 8(t-3)^6 \mid t-3 \mid + (t-4)^6 \mid t-4 \mid]. \end{aligned} \qquad (3.2)$$

This function is a B-spline, symmetric with respect to $t = 0$ and its support is defined by an interval [–4, 4] (Figure 2).

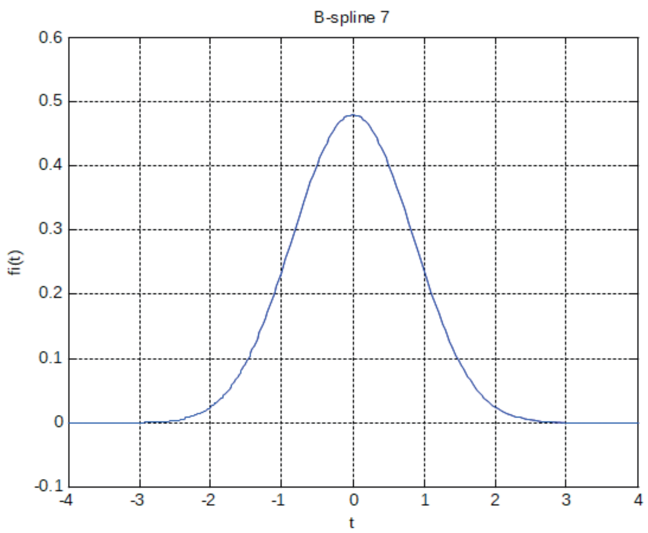



*Figure 2. B-spline of the seventh order $\varphi(t) = \varphi_{0.8}(t+4)$.*

Let us use the following notation system:

$$\begin{gathered} \varphi_1(t) = \varphi(t+3),\ \varphi_2(t) = \varphi(t+2), \\ \varphi_3(t) = \varphi(t+1),\ \varphi_4(t) = \varphi(t), \\ \varphi_5(t) = \varphi(t-1), \\ \varphi_6(t) = \varphi(t-2),\ \varphi_7(t) = \varphi(t-3), \\ \varphi_8(t) = \varphi(t-4),\ 0 \le t \le 1. \end{gathered} \qquad (3.3)$$

We represent the unknown deflection function in the form

$$y(x) = w(t) = \sum_{k=1}^{N} \alpha_k \varphi_k(t),$$
$$x_{1(i_e)} \le x \le x_{6(i_e)},\ 0 \le t \le 1. \qquad (3.4)$$

We can substitute (3.4) into (1.3), taking into account relations (3.1).

$$\begin{aligned} \Phi_{i_e}(y) &= \frac{1}{2} \int_{x_1(i_e)}^{x_6(i_e)} \left( EJ \left( \frac{d^2 y}{dx^2} \right)^2 + \beta y^2 \right) dx - \int_{x_1(i_e)}^{x_6(i_e)} q y dx = \\ &= \frac{1}{2} \int_0^1 \left( \frac{EJ}{h_e^3} (w'')^2 + \beta h_e w^2 \right) dt - \int_0^1 h_e q w dt = \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \times \\ &\times \int_0^1 \left( \frac{EJ}{h_e^3} \left( \varphi''_i(t) \varphi''_j(t) \right) + \beta h_e \left( \varphi_i(t) \varphi_j(t) \right) \right) dt - \\ &- \sum_{i=1}^{N} \alpha_i \int_0^1 h_e q \varphi_i(t) dt = \\ &= \frac{1}{2} (K_\alpha^{i_e} \bar{\alpha}, \bar{\alpha}) - (\bar{R}_\alpha^{i_e}, \bar{\alpha}) = \Phi_\alpha(\bar{\alpha}), \end{aligned} \qquad (3.5)$$

where we have

$$K_\alpha^{i_e}(i,j) = \int_0^1 \left( \frac{EJ}{h_e^3} \left( \varphi''_i(t) \varphi''_j(t) \right) + \beta h_e \left( \varphi_i(t) \varphi_j(t) \right) \right) dt;$$

$$R_\alpha^{i_e}(i) = \int_0^1 \left( h_e q(t) \varphi_i(t) \right) dt.$$

Let's define the parameters through the nodal unknowns on the element:

$$\begin{cases} y_1 = w(0) = \sum_{k=1}^{N} \alpha_k \varphi_k(0) \\ \frac{dy_1}{dx} = \frac{1}{h_e} w'(0) = \frac{1}{h_e} \sum_{k=1}^{N} \alpha_k \varphi'_k(0) \\ y_2 = w(0.2) = \sum_{k=1}^{N} \alpha_k \varphi_k(0.2) \\ y_3 = w(0.4) = \sum_{k=1}^{N} \alpha_k \varphi_k(0.4) \\ y_4 = w(0.6) = \sum_{k=1}^{N} \alpha_k \varphi_k(0.6) \\ y_5 = w(0.8) = \sum_{k=1}^{N} \alpha_k \varphi_k(0.8) \\ y_6 = w(1) = \sum_{k=1}^{N} \alpha_k \varphi_k(1) \\ \frac{dy_6}{dx} = \frac{1}{h_e} w'(1) = \frac{1}{h_e} \sum_{k=1}^{N} \alpha_k \varphi'_k(1) \end{cases}$$

Therefor we have

$$\overline{y}^{i_e} = T\overline{\alpha}\,, \qquad (3.6)$$

where

$$\overline{y}^{i_e} = [y_1 \ \ \frac{dy_1}{dx} \ \ y_2 \ \ y_3 \ \ y_4 \ \ y_5 \ \ y_6 \ \ \frac{dy_6}{dx}]^{\mathrm{T}}\,;$$

$$\overline{\alpha} = [\alpha_1 \ \ \alpha_2 \ \ \alpha_3 \ \ \alpha_4 \ \ \alpha_5 \ \ \alpha_6 \ \ \alpha_7 \ \ \alpha_8]^{\mathrm{T}}\,;$$

$$D = diag(1 \ \ 1/h_e \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1/h_e)\,;$$

$$T = D\begin{bmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) & \varphi_4(0) & \varphi_5(0) & \varphi_6(0) & \varphi_7(0) & \varphi_8(0) \\ \varphi'_1(0) & \varphi'_2(0) & \varphi'_3(0) & \varphi'_4(0) & \varphi'_5(0) & \varphi'_6(0) & \varphi'_7(0) & \varphi'_8(0) \\ \varphi_1(0.2) & \varphi_2(0.2) & \varphi_3(0.2) & \varphi_4(0.2) & \varphi_5(0.2) & \varphi_6(0.2) & \varphi_7(0.2) & \varphi_8(0.2) \\ \varphi_1(0.4) & \varphi_2(0.4) & \varphi_3(0.4) & \varphi_4(0.4) & \varphi_5(0.4) & \varphi_6(0.4) & \varphi_7(0.4) & \varphi_8(0.4) \\ \varphi_1(0.6) & \varphi_2(0.6) & \varphi_3(0.6) & \varphi_4(0.6) & \varphi_5(0.6) & \varphi_6(0.6) & \varphi_7(0.6) & \varphi_8(0.6) \\ \varphi_1(0.8) & \varphi_2(0.8) & \varphi_3(0.8) & \varphi_4(0.8) & \varphi_5(0.8) & \varphi_6(0.8) & \varphi_7(0.8) & \varphi_8(0.8) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) & \varphi_5(1) & \varphi_6(1) & \varphi_7(1) & \varphi_8(1) \\ \varphi'_1(1) & \varphi'_2(1) & \varphi'_3(1) & \varphi'_4(1) & \varphi'_5(1) & \varphi'_6(1) & \varphi'_7(1) & \varphi'_8(1) \end{bmatrix}$$

Therefor we have

$$\overline{\alpha} = T^{-1}\overline{y}^{i_e}\,. \qquad (3.7)$$

Substituting (3.7) into $\Phi_\alpha(\overline{\alpha})$, we obtain

$$\begin{aligned} \Phi_\alpha(\overline{\alpha}) &= \\ &= \frac{1}{2}(K_\alpha^{i_e} T^{-1}\overline{y}^{i_e}, T^{-1}\overline{y}^{i_e}) - (\overline{R}_\alpha^{i_e}, T^{-1}\overline{y}^{i_e}) = \\ &= \frac{1}{2}((T^{-1})^{\mathrm{T}} K_\alpha^{i_e} T^{-1}\overline{y}^{i_e}, \overline{y}^{i_e}) - ((T^{-1})^{\mathrm{T}}\overline{R}_\alpha^{i_e}, \overline{y}^{i_e}) = \\ &= \frac{1}{2}(K^{i_e}\overline{y}^{i_e}, \overline{y}^{i_e}) - (\overline{R}^{i_e}, \overline{y}^{i_e}) = \Phi_{i_e}(\overline{y}^{i_e}), \end{aligned} \qquad (3.8)$$

where

$$K^{i_e} = (T^{-1})^{\mathrm{T}} K_\alpha^{i_e} T^{-1}$$

is the local stiffness matrix;

$$\overline{R}^{i_e} = (T^{-1})^{\mathrm{T}} \overline{R}_\alpha^{i_e}$$

is the local load vector.

.

## 4. INFORMATION ABOUT NUMERICAL IMPLEMENTATION

The presented algorithm can be implemented using MATLAB tools. The MATLAB system has convenient functions for working with polynomials. Moreover, the main parameter of these functions is the vector of coefficients of the polynomial. To determine the coefficients of basic polynomials $\varphi_k$ on an interval [0  1], we can firstly determine their values at eight points of the interval $t = [t_1, t_2, ..., t_8]$, $t_i \in [0 \;\; 1]$, $i = 1, 2, ..., 8$;

$$F_k(i) = \varphi_k(t_i),\ i = 1, 2, ..., 8,\quad k = 1, 2, ..., 8 .$$

Then, using the `polyfit` function, we define their coefficient vector:

```
pk=polyfit(t,Fk,7)
```

This function is used to determine the coefficients of the optimal polynomial using the least squares method. In the considering case, we are looking for a polynomial of the 7th degree (i.e. we have to define 8 coefficients of polynomial, according to its 8 values), therefore, we get a polynomial passing through the given values.
In order to calculate the derivatives we can sequentially use the `polyder` function:

```
dpk=polyder(pk)
```
is the vector of coefficients $\varphi'_k$;
```
d2pk=polyder(dpk)
```
is the vector of coefficients $\varphi''_k$.

In order to calculate the product of polynomials we can use the `conv` function:

```
pij=conv(pi,pj)
```
is the vector of coefficients $\varphi_i\varphi_j$;
```
d2pij=conv(d2pi, d2pj)
```
is the vector of coefficients $\varphi''_i\varphi''_j$.

In order to calculate the antiderivative of a polynomial we can use the `polyint` function:

```
Pi=polyint(pi)
```
is the vector of coefficients $\int \varphi_i dt$;
```
Pij=polyint(pij)
```
is the vector of coefficients $\int \varphi_i\varphi_j dt$;
```
d2Pij=polyint(d2pij)
```
is the vector of coefficients $\int \varphi''_i\varphi''_j dt$.

Then the calculation (formula (3.5))

$$K_\alpha^{i_e}(i, j) = \int_0^1 \left( \frac{EJ}{h_e^3}\left(\varphi''_i(t)\varphi''_j(t)\right) + \beta h_e\left(\varphi_i(t)\varphi_j(t)\right)\right) dt .$$

can be summarized as follows:

$K_\alpha^{i_e}(i, j) = \frac{EJ}{h_e^3}$`( polyval(d2Pij,1)-polyval(d2Pij,0))+` $+\beta h_e$`( polyval(Pij,1)-polyval(Pij,0)),`

where the function `polyval (p, t)` allows researcher to calculate the values of a polynomial with a vector of coefficients $p$ at a given point $t$.
As for the calculation (see (3.5)),

$$R_\alpha^{i_e}(i) = \int_0^1 \left(h_e q(t)\varphi_i(t)\right) dt$$

here, for example, the following options are possible:
– point load setting (using delta functions);
– setting the load averaged on the element,

$R_\alpha^{i_e}(i) = h_e q_{ie}$`(polyval (Pi, 1) - - polyval (Pi, 0))`

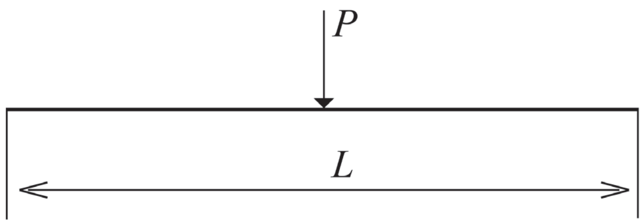



*Figure 3. Example of analysis.*

If $q$ is represented by a polynomial, then, as in the case of calculating the elements of a local matrix $K_\alpha^{i_e}$, here researcher can use the function of multiplying polynomials `conv` followed by determining the antiderivative of the product using the `polyint` functions and calculating the definite integral using the `polyval` function.

## 5. EXAMPLE OF ANALYSIS

As a model example let us consider a beam on an elastic foundation with the following parameters:

$$q(x) = P\delta(x - \frac{L}{2}), \quad P = 100 \text{ kN}$$

is load given at the midpoint (Figure 3);

$$L = 8m;\ h_b = 1.3m;\ b_b = 1m;$$
$$E = 2560 \cdot 10^4\ kN/m^2;\ k = 75 \cdot 10^3 kN/m^3.$$

In this case we should consider the following boundary conditions:

$$\begin{cases} y(0) = y''(0) = 0 \\ y(L) = y''(L) = 0 \end{cases}$$

– the beam is hingedly supported on both sides (the first case);

$$\begin{cases} y(0) = y'(0) = 0 \\ y(L) = y'(L) = 0 \end{cases}$$

– the beam is rigidly fixed on both sides (the second case);

$$\begin{cases} y(0) = y''(0) = 0 \\ y'''(L) = y''(L) = 0 \end{cases}$$

– the beam is hingedly supported on the left end, the right end is free (the third case);

$$\begin{cases} y(0) = y'(0) = 0 \\ y'''(L) = y''(L) = 0 \end{cases}$$

the beam is rigidly fixed to the left end, the right end is free (the fourth case).
Let us set $N_e = 4$ (the number of elements).
Then we have

$$N_g = N_p + 2N_b = 4 \cdot (5 - 1) + 2 \cdot (4 + 1) = 26;$$

is the total number of unknowns;

$$h_e = L / N_e = 8 / 4 = 2$$

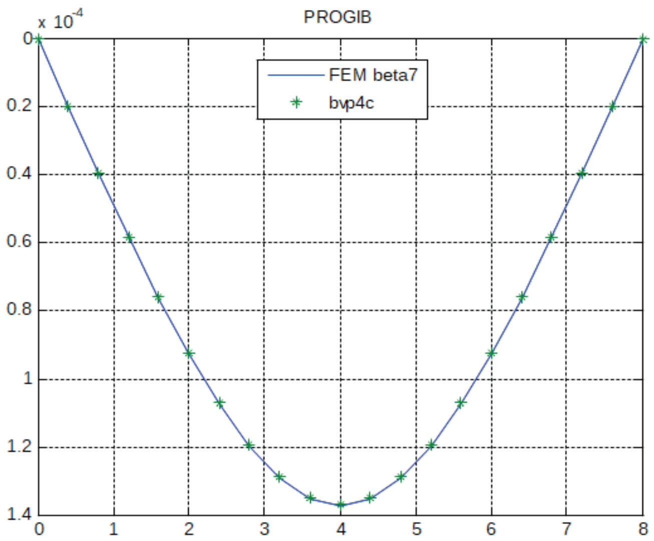
is the length of the element;

$$h_p = h_e / 5 = 2 / 5 = 0.4$$

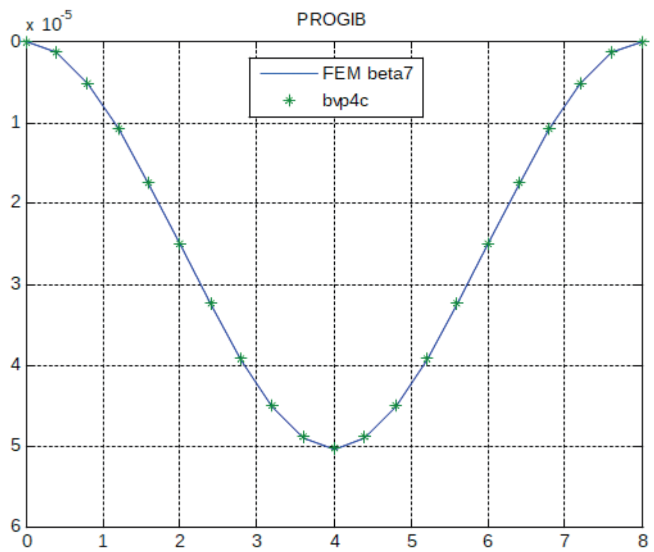
is the step between the coordinates of the nodes;

$$N_x = L / h_p + 1 = 8 / 0.4 + 1 = 21$$

is the total number of nodes.
Several results of analysis are presented at Figures 4, 5, 6 and 7.




*Figures 4. Comparison of results for the first case.*




*Figures 5. Comparison of results for the second case.*

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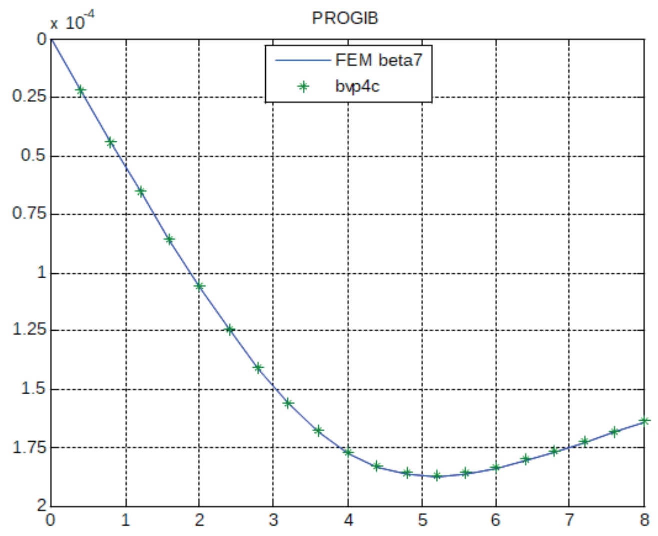
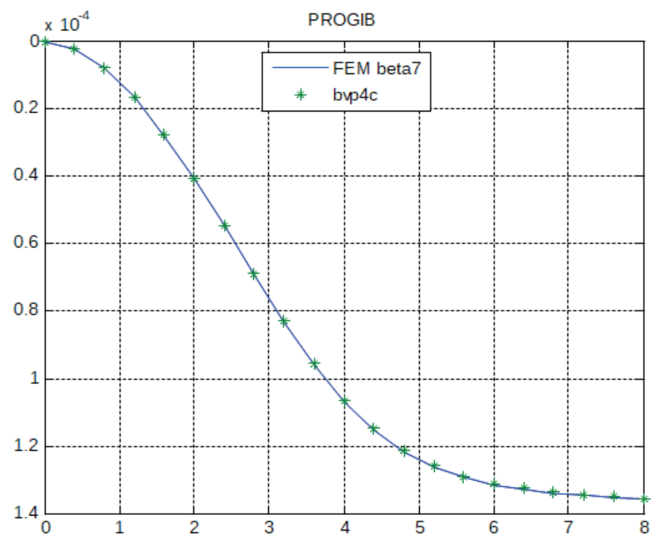



*Figures 6. Comparison of results for the third case.*




*Figures 7. Comparison of results for the fourth case.*

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# ANALYTICAL CALCULATION OF DEFLECTION OF A MULTI-LATTICE TRUSS WITH AN ARBITRARY NUMBER OF PANELS

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**Abstract**: The scheme of a planar externally statically indeterminate truss with four supports is proposed. In analytical form, for several types of loads, the problem of forces in the rods and deflection of the structure is solved, depending on the number of panels, the size and intensity of the load. The solution uses the Maple computer mathematics system. The deflection at Midspan is determined using Maxwell – Mohr's formula, the forces in the rods – the method of cutting out nodes from the system of equilibrium equations for all nodes, which includes four reactions of the supports. By induction, a series of solutions for trusses with a consistently increasing number of panels is generalized to an arbitrary number of panels. For the elements of the sequences of coefficients are developed and are solved by homogeneous linear recurrence equations. The resulting formulas for the deflection of the structure under various loads have the form of polynomials in the number of panels. A linear asymptotic solution for the number of panels is found. The kinematic degeneration of the structure and the distribution of node speeds corresponding to this case were found. The dependences of the reaction of supports and forces in the most compressed and stretched rods on the number of panels are determined.

**Keywords**: truss, deflection, induction, Mohrs' integral, Maple, kinematic degeneration

# АНАЛИТИЧЕСКИЙ РАСЧЕТ ПРОГИБА МНОГОРЕШЕТЧАТОЙ ФЕРМЫ С ПРОИЗВОЛЬНЫМ ЧИСЛОМ ПАНЕЛЕЙ

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**Аннотация**: Предлагается схема плоской внешне статически неопределимой фермы с четырьмя опорами. В аналитической форме для нескольких видов нагрузок решается задача об усилиях в стержнях и прогибе конструкции в зависимости от числа панелей, размеров и интенсивности нагрузки. Для решения используется система компьютерной математики Maple. Прогиб в середине пролета определяется по формуле Максвелла – Мора, усилия в стержнях – методом вырезания узлов из системы уравнений равновесия всех узлов, в которую включаются и четыре реакции опор. Методом индукции серия решений для ферм с последовательно увеличивающимся числом панелей обобщается на произвольное число панелей. Для элементов последовательностей коэффициентов составляются и решаются однородные линейные рекуррентные уравнения. Полученные формулы для прогиба конструкции при различных нагружениях имеют вид полиномов по числу панелей. Найдена линейная асимптотика решения по числу панелей. Обнаружено кинематическое вырождение конструкции и распределение скоростей узлов, соответствующее этому случаю. Определены зависимости реакций опор и усилий в наиболее сжатых и растянутых стержнях от числа панелей.

**Ключевые слова**: ферма, прогиб, индукция, интеграл Мора, Maple, кинематическое вырождение

## INTRODUCTION

The calculation of rod structures is usually performed in numerical packages based on the finite element method [1–4]. The usual solution of the mechanics problem, performed not in a numerical package, but in a system of symbolic mathematics, without changing the basic equations and calculation scheme, gives an analytical solution to the problem in the form of a formula. In the years when computer mathematics systems first appeared, this caused the optimism of calculators who know the importance of analytical solutions. However, almost immediately, many on this path encountered two obstacles. First, most of the resulting formulas were so complex that it was not only impossible to use them, but even difficult to view them, since their listing took up several pages. The second disadvantage of the solutions obtained in this way is that the range of applicability of the obtained formulas (if they are obtained in a relatively compact form) is usually not wide. Among the parameters of formulas, you can easily enter the size of the calculated object, elastic or rheological properties of the material, and the intensity of a certain load. In order to use a formula with a different number of structural elements, such as rods or panels, if you are talking about trusses, you must output a formula that is intended for this number. If overcoming the first disadvantage of analytical solutions associated with their bulkiness is possible with some skill in working with simplification operators included in computer mathematics systems, the second disadvantage can be overcome using the induction method [5]. The induction method is applicable for regular constructions that have periodicity cells of the structure. Solutions are known for a number of planar [6–13] and spatial [14] statically definable trusses. The significance of regular statically definable schemes was first evaluated by Hutchinson R. G., Fleck N. A., Zok F. W., Latture R. M., Begley M. R. [15–17]. Monographs [18,19] are devoted to such schemes and methods of their calculation. The reference book [20] contains more than 70 schemes of planar trusses and formulas for calculating deflection and forces in rods critical to stability and strength. Tinkov D.V. [21] and Osadchenko N.V. [22] provides an overviews of some analytical solutions for planar trusses.

## MATERIALS AND METHODS

### The geometry of the truss. The case of variability of the design

Let's consider a symmetrical lattice truss of beam type with 2n panels, counting the elements of the upper belt with length a (Fig. 1). In its middle part, the lower belt is slightly raised. Due to the four supports, the truss is externally statically indeterminate. The reactions of the supports of such a truss can only be calculated from the joint solution of the system of equilibrium equations of all nodes simultaneously with the forces in the rods. The truss contains $m = 8n + 24$ rods, including six rods that model movable and fixed supports.

We will calculate the forces in the rods using the program [6-13], compiled in the language of the Maple system, which is close to the Pascal language. The program includes the coordinates of the joints and the structure of the connection of the rods. The matrix of a system of equations consists of the guiding cosines of forces. The vector of the right part of the system of equilibrium equations includes loads on nodes. At the same time, in the first test calculations, it was noticed that for trusses with an even number of panels $n$ in half the span, the matrix determinant degenerates, which indicates the instantaneous variability of the system [20, 23]. Note that calculations in numerical form hid the fact that the determinant turned to zero for the error of the calculation, and only analytical (or integer) calculation clearly gave out this dangerous feature of the construction under consideration. A picture of the distribution of possible velocities of nodes is obtained (Fig. 2), confirming the kinematic variability of the truss.

The following velocity ratios are obtained from considering the positions of the instantaneous velocity centers of individual rods: $u'/h = v/a$, $2u/c = v/a$ where $c = \sqrt{a^2 + h^2}$. Most of the truss joints and supports remain stationary.

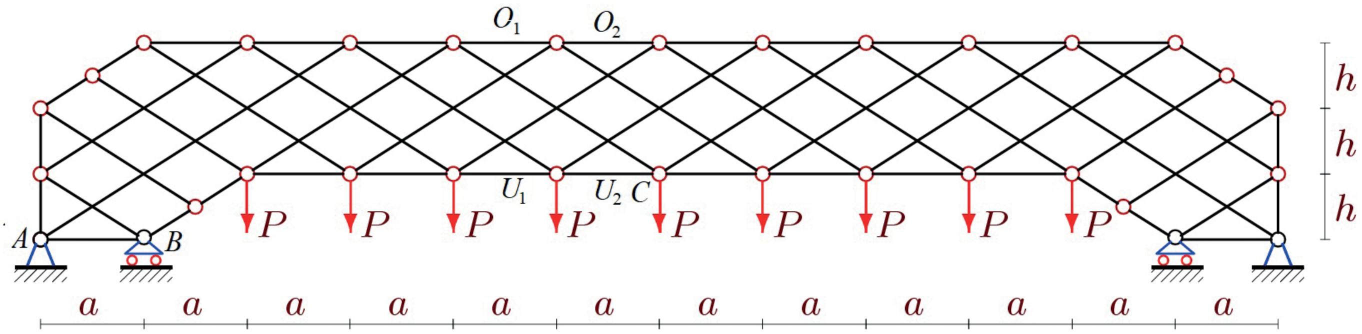


*Figure 1. The load on the bottom belt, n = 5*

## RESULTS

### The Forces In The Rods

The distribution of forces in the truss rods at $a$ = 4 m, $h$ = 3 m from the action of the load applied to the nodes of the lower belt, obtained from the numerical calculation data (Fig. 3), shows that the upper belt is partially compressed, the lower one is stretched in its central part . Compressed elements are highlighted in blue, stretched elements in red, and unloaded ones in black. The thickness of the lines is proportional to the modulus of force. The efforts are related to the value of force $P$. With an increase in the number of panels, the stretched zone in the lower zone naturally expands. It should be noted that the most compressed rods are not in the middle of the span. Using the induction method, one can obtain analytical expressions of the reactions of supports and forces in some rods of the truss (marked in Fig. 1). We have the following expressions for the reactions of supports:

$$Y_A = 2P(k-1),\ Y_B = P/2,$$
$$X_A = P(4k-3)a/(2h).$$

Forces in the middle of the upper belt:

$$O_1 = -P(4k^2 - 2k(-1)^k - 4k + (-1)^k - 1)a/(4h),$$
$$O_2 = -P(4k^2 + 2k(-1)^k - 4k + (-1)^k + 1)a/(4h).$$

Forces in the lower belt:

$$U_1 = P(4k^2 + 2k(-1)^k - 12k - (-1)^k + 1)a/(4h),$$
$$U_2 = P(4k^2 - 2k(-1)^k - 12k - (-1)^k + 3)a/(4h)$$

### Deflection

Truss deflection (vertical displacement of the middle node $C$ from the lower belt) it is determined by the Maxwell-Mohr's formula $\Delta = \sum_{i-1}^{m-6} S_i^{(P)} S_i^{(1)} l_i/(EF)$ , where the sum is calculated only for deformable truss rods. It is indicated: $S_i^{(1)}$ – forces from the unit force applied to the lower belt, $S_i^{(P)}$ – forces in the rods from a given load, $l_i$ – the length of the rods, $EF$ — their stiffness.

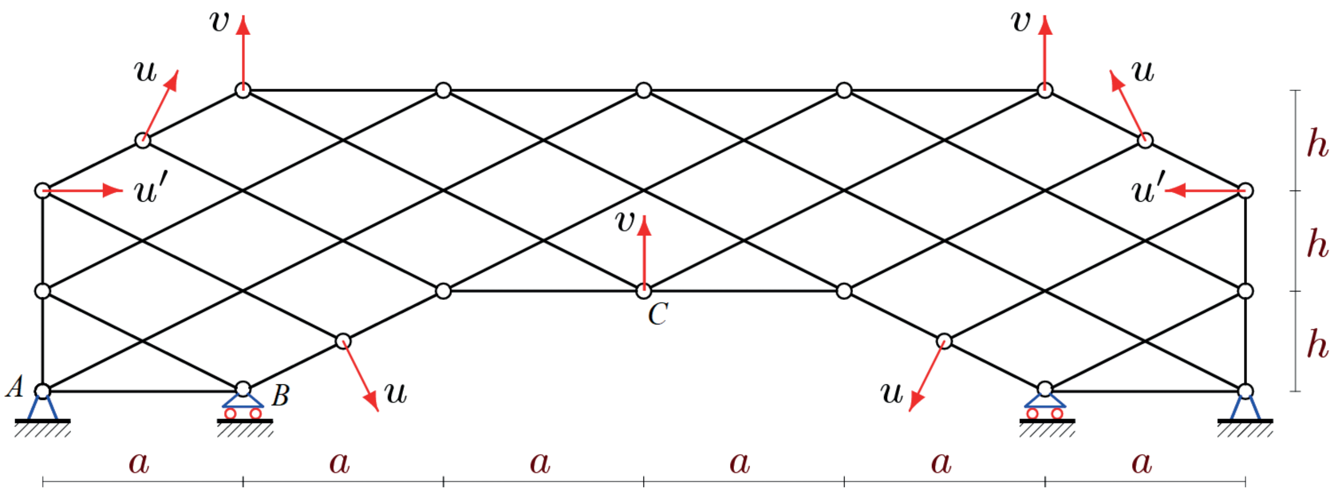


*Figure 2. Velocities distribution of an instantaneous variable truss, n =2*

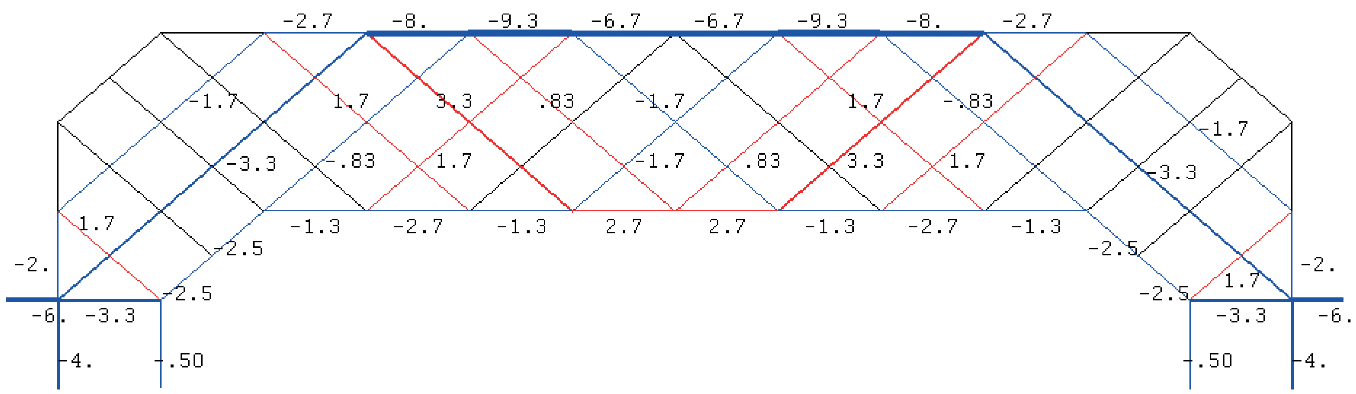


*Figure 3. The distribution of forces in the truss, n =5*

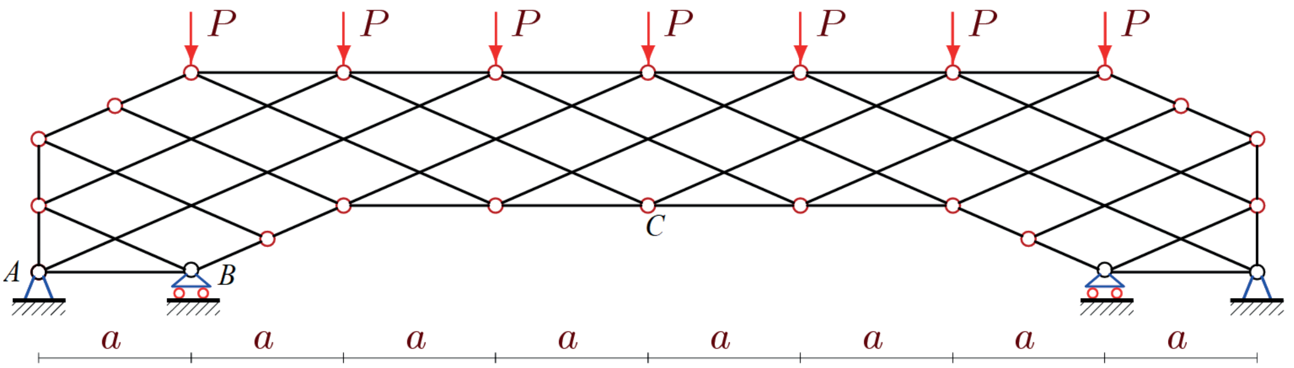


*Figure 4. The load on the upper belt, n = 3*

Let's consider the case of a uniform load on the nodes of the upper belt (Fig. 1). Regardless of the number of panels, the deflection has the form:

$$\Delta = P(C_1a^3 + C_2c^3 + C_3h^3) / (h^2EF) . \qquad (1)$$

Coefficients for size degrees depend only on the number of panels. We consider odd numbers for which the determinant of the system of linear equations of equilibrium of nodes does not turn to zero. To determine these dependencies, you need to calculate a number of trusses with a consistently increasing number of panels and find common members of the sequences. To determine the coefficient $C_1$, it was necessary to calculate 18 trusses with the number k =1,..., 18 and get the sequence 1/2, 19/2, 53/2, 383/2,..., 292115/2.

First the rgf_findrecur operator returns a linear homogeneous recurrent equation for elements in the sequence:

$$C_{1,k} = C_{1,k-1} + 4C_{1,k-2} - 4C_{1,k-3} - 6C_{1,k-4} +$$
$$+ 6C_{1,k-5} + 4C_{1,k-6} - 4C_{1,k-7} - C_{1,k-8} + C_{1,k-9} .$$

Then the General term of this sequence, as a solution of the recurrent equation, gives the rsolve operator:

$$C_1 = (20k^4 + 16k^3(-1)^k - 80k^3 - 48k^2(-1)^k +$$
$$+ 130k^2 + 50k(-1)^k - 58k - 9(-1)^k + 3) / 12 .$$

Other coefficients are obtained in the same way:

$$C_2 = (k^2 + k(-1)^k - (-1)^k) / 2$$
$$C_3 = (k - 1)(1 + (-1)^k) .$$

Expression (1) with the found dependencies $C_i = C_i(k)$, $i = 1,2,3$ is the solution to the problem.

The used algorithm for output of calculation formulas can be easily adjusted to other loads. Consider the load on the upper belt of the truss (Fig. 4).

The coefficients in (1) in this case have the form:

$$C_1 = (20k^4 + 16k^3(-1)^k - 80k^3 - 48k^2(-1)^k +$$
$$+ 130k^2 + 50k(-1)^k - 70k - 15(-1)^k + 9) / 12 ,$$
$$C_2 = k(k + (-1)^k) / 2 ,$$
$$C_3 = k(1 + (-1)^k) .$$

In the case of loading the truss with a single force applied to the hinge C in the middle of the lower belt, the problem is solved somewhat easier. The coefficients in expression (1) have a lower degree:

$$C_1 = (4k^3 + 6k^2(-1)^k - 12k^2 - 12k(-1)^k +$$
$$+ 20k + 9(-1)^k - 6) / 6 ,$$
$$C_2 = k + (-1)^k / 2 ,$$
$$C_3 = 1 + (-1)^k .$$

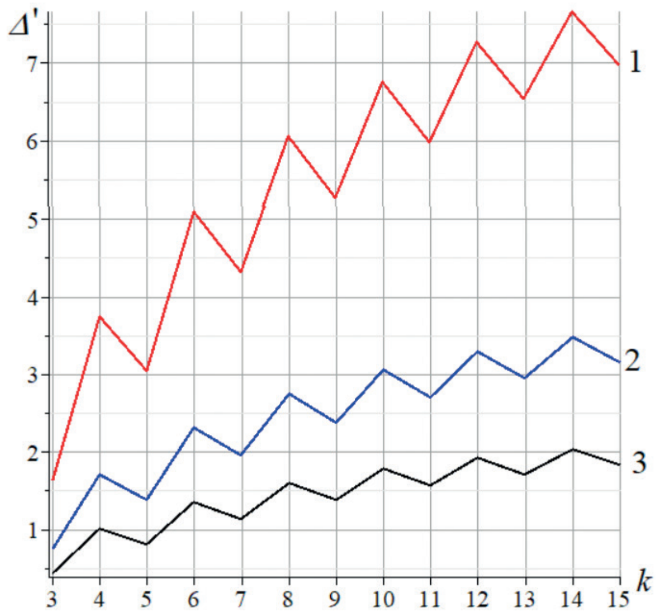


*Figure 5. Dependence of the deflection on the number of panels*
*1 — h = 2 m, 2 — h = 3 m, 3 — h = 3 m*

The proposed truss scheme has a number of features that are most conveniently traced by example. Consider a truss of a given length $L = 2(n+1)a$ loaded in the lower zone. We also fix the total load on the truss: $P_{sum} = (2n-1)P$. We introduce the dimensionless relative deflection: $\Delta' = \Delta EF / (P_{sum} \cdot L)$. Figure 5 at $L$ = 80m shows the dependence of the deflection on the number of panels at various values of the height of the truss. Dependencies have a pronounced spasmodic character. The jumps are especially large at low altitudes and small numbers $k$. As $k$ increases, the curves smooth out, tending to some oblique asymptote. Using Maple, the slope can be calculated:

$$\lim_{n\to\infty} \Delta'/k = h/(8L) .$$

The angle of inclination is positive, therefore, with an increase in the number of panels with a simultaneous decrease in their length, due to the accepted assumption that the total length of the truss is constant, the relative deflection increases on average (including jumps).

## CONCLUSIONS

Two main conclusions can be drawn. First, the analytical solution for the proposed truss scheme has a simple form. it is valid for an arbitrary number of panels, including a very large number, i.e. precisely in cases when numerical methods can accumulate rounding errors and require significant counting time. Second, the discovery of an unexpected case of kinematic variability should serve as a warning for designers of new schemes, where the degeneracy of the determinant of the system of equations of equality may be hidden behind rounding of intermediate data. Noticeable jumps in the deflection dependence on the number of panels are the basis for optimal selection of the number of panels. Reducing or increasing the number of panels by one can change the stiffness from 10 % to 100% depending on the number of panels. The linear combination of solutions obtained for three types of loads allows us to solve a wide range of problems for truss of the considered type in analytical form.

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# THE PROBLEMS OF COMPUTATION OF COMBINED PLATES WITH PIECEWISE VARIABLE THICKNESS. SOLUTIONS IN ORTHOGONAL POLYNOMIALS

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**Abstract**: The work is devoted to the analytical simulation of the combined plates calculation. The mentioned plates have the circular form and they consist of separate parts with different laws of thickness variation. These sections may be made from the same or from different materials. The material can be homogeneous or nonhomogeneous, isotropic or anisotropic. In the places of the separate sections conjugation the construction's thickness can be continuous or discontinuous. The construction under study is subjected to an action of bending loads. Below the analytical method for the similar constructions' computation is suggested. This method is based on the use of the theory of the special functions, in particular, Lagerr's orthogonal polynomials.

**Keywords**: combined constructions, piecewise variable thickness, Lagerr's orthogonal polynomials.

# ПРОБЛЕМЫ РАСЧЕТА КОМБИНИРОВАННЫХ ПЛАСТИН КУСОЧНО-ПЕРЕМЕННОЙ ТОЛЩИНЫ. РЕШЕНИЯ В КЛАССИЧЕСКИХ ОРТОГОНАЛЬНЫХ МНОГОЧЛЕНАХ

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**Аннотация**: Работа посвящена аналитическому моделированию проблем расчета комбинированных пластин, имеющих в плане круговую форму и состоящих из отдельных участков, в которых толщина изменяется по различным законам. Эти отдельные участки могут быть сделаны как из одного и того же, так и из различных материалов, которые могут обладать свойствами однородности или неоднородности; быть изотропными или анизотропными. В местах стыков отдельных участков толщина конструкции может быть или непрерывной, или иметь разрыв непрерывности. Изучаемые конструкции работают на изгиб. Ниже предлагается аналитическая методика расчета подобных конструкций, связанная с использованием классических ортогональных многочленов, в частности, многочленов Лагерра.

**Ключевые слова**: комбинированные конструкции, кусочно-переменная толщина, многочлены Лагерра.

## 1. INTRODUCTION

The plates having a circular form and consisting of two or a few parts with various laws of thickness variation are under consideration. Such plates occur as constructive elements in modern buildings and structures. Their separate parts may be made from the same or different materials. These materials can be homogeneous or inhomogeneous, isotropic or anisotropic. In the places of the sections conjugation the plate's thickness can be continuous or it has a discontinuity. The analytical methods of the such construction computation, specifically connected with the theory of the special functions, are not yet developed. The work [1] is to be mentioned. In this work the foundation slab, resting on an elastic subgrade, was under consideration. The plate's inner part has variable thickness, the outer part has the constant thickness. The solutions were received in terms of Bessel functions. The present work considers the bending of the combined plate with the piecewise variable thickness. The solutions are obtained in the closed form in terms of the Lagerr’s orthogonal polynomials and the confluent functions.

## 2. THE STATEMENT OF THE PROBLEM

The works, in which to the circular plates of variable and constant thickness analysis the theory of the special functions is used, are known in literature, for example [2], [3], [4].

Let us go to the consideration of the combined plates which were described above (Fig.1).

The differential equation, describing the symmetric bending of the circular orthotropic plate with the varying thickness, has the form [3], [4]:

$$r^2\frac{\partial^2\vartheta}{\partial r^2}+r\left(1+\frac{r}{D}\frac{dD}{dr}\right)\frac{d\vartheta}{dr}+\left(\frac{\sigma r}{D}\frac{dD}{dr}-n^2\right)\vartheta+ \\ +\frac{r}{Dn_2}\left(\int q_Z r dr-C\right)=0, \quad (1)$$

here $\vartheta=-\frac{dw}{dr}$, σ is the Poisson’s ratio, the parameters $n^2=n_1n_2$ are determined by the following expressions:

$$E_r=\frac{E}{n_2},\ E_\theta=En_2,\ \sigma_r=\frac{\sigma}{n^2},\ \sigma_\theta=\sigma\ . \quad (2)$$

For isotropic plate $n_1 = n_2 = 1$.

Let us write:

$$-\int q_z r dr + C = Q_r r. \quad (3)$$

The stresses in the orthotropic circular plate of variable thickness are determined from the following expressions:

$$M_r=Dn_2\left(\frac{d\vartheta}{dr}+\frac{\sigma}{r}\vartheta\right),$$

$$Q_r=Dn_2\left(\frac{d^2\vartheta}{dr^2}+\frac{1}{r}\frac{d\vartheta}{dr}-\frac{n^2}{r^2}\vartheta\right)+ \\ +\frac{dD}{dr}n_2\left(\frac{d\vartheta}{dr}+\frac{\sigma}{r}\vartheta\right). \quad (4)$$

Introducing the independent argument:

$$x=\left(\frac{r}{r_0}\right)^{\alpha_0}, \quad (5)$$

where $\alpha_0$, $r_0$ – are the constants.

Substituting (5) into (1) we get, assuming $q_Z = 0$:

$$\frac{d^2\vartheta}{dx^2}+\left(\frac{1}{x}+\frac{1}{D}\frac{dD}{dx}\right)\frac{d\vartheta}{dx}+ \\ +\frac{1}{\alpha_0 x}\left(\frac{\sigma}{D}\frac{dD}{dx}-\frac{n^2}{\alpha_0 x}\right)\vartheta-\frac{Cr_0 x^{-2+\frac{1}{\alpha_0}}}{Dn_2\alpha_0^2}=0. \quad (6)$$

We consider the cases of symmetric bending of orthotropic circular plates of variable thickness which allow receiving the solution in terms of Lagerr’s orthogonal polynomials. Let us write the differential equation for Lagerr’s polynomials [5]:

$$y''+\frac{\alpha+1-x}{x}y'+\frac{m}{x}y=0. \quad (7)$$

As the result we receive that the sought solution occur for the following law of flexural rigidity variation:

$$D = D_0 x^{\alpha} e^{-x}, \quad (8)$$

where

$$\alpha = -mn^2/\sigma^2,\ \alpha_0 = -\sigma/m. \quad (9)$$

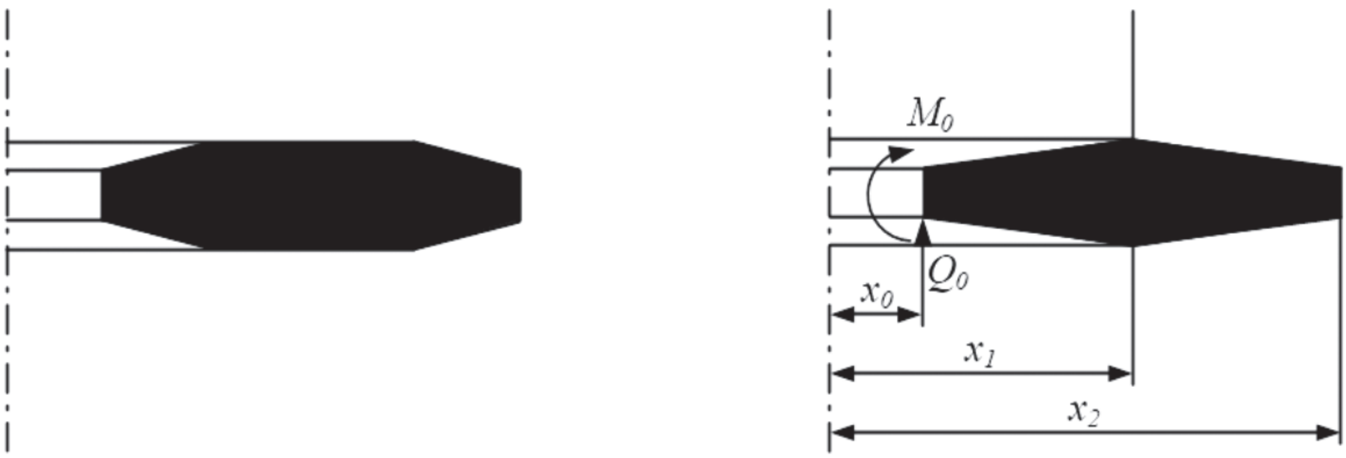


*Figure 1. The combined plates with the piecewise variable thickness*

The general solution of the homogeneous equation, corresponding to (6), is

$$\vartheta = AL_m^{\alpha}(x) + Bx^{-\alpha}\,{}_1F_1(-m-\alpha;1-\alpha;x). \quad (10)$$

In the similar way we can get the solutions in terms of different polynomials, for example in Chebyshev or Hermite polynomials. However these laws have more restricted domain of definition than (8).
The following law of thickness variation, corresponding to the flexural rigidity (8), is

$$h = h_0 x^{\alpha/3} e^{-x/3}. \quad (11)$$

The set of curves, corresponding to the profiles (11), can be built. In this case it must be taken into account that the Poisson's ratio $\sigma$ is limited (9).

## 3. THE COMBINED PLATE

The combined plate with piecewise thickness variation is under consideration. The proposed method will be shown on the example of the combined plate consisting of two parts. However the suggested method can be applied for combined plates, consisting of several parts, analysis. Let us assume that in our example the plate's thickness is continuous in the place of joint (Fig.2).

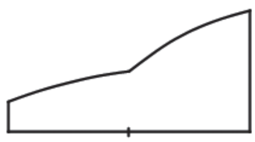


*Figure 2. The combined plate consisting of two parts*

The special auxiliary functions are introduced for the realization of the separate parts joint and for consideration of the action of discontinuous loads, distributed along the circles non-coinciding with the plate's contour.
First we shall write the wronskian for the solutions (10):

$$W(x) = \binom{m+\alpha}{m} \alpha x^{-\alpha-1} e^{-x}. \quad (12)$$

Next the Cauchy functions for the solutions (10) $Y_1(x_1; x)$, $Y_2(x_1; x)$ are to be obtained. The indicated functions are defined by the expressions:

$$\begin{aligned}
Y_1(x_1;x) &= \binom{m+\alpha}{m}^{-1} \alpha^{-1} e^{-x_1} x^{\alpha+1} \times \\
&\times \{[\alpha x_1^{-\alpha-1}\,{}_1F_1(-m-\alpha;1-\alpha;x_1) + x_1^{-\alpha} \times \\
&\times \frac{m+\alpha}{1-\alpha}\,{}_1F_1(-m-\alpha+1;2-\alpha;x_1)]L_m^{\alpha}(x) - \\
&- \left[\frac{m}{x_1} L_m^{\alpha}(x_1) - \frac{m+\alpha}{x_1} L_{m-1}^{\alpha}(x_1)\right] \times \\
&\times x^{-\alpha}\,{}_1F_1(-m-\alpha;1-\alpha;x)\}, \\
Y_2(x_1;x) &= \binom{m+\alpha}{m}^{-1} \alpha^{-1} e^{-x_1} x_1^{\alpha+1} \times \\
&\times \{-L_m^{\alpha}(x_1) x^{\alpha}\,{}_1F_1(-m-\alpha;1-\alpha;x) + \\
&+ x_1^{\alpha}\,{}_1F_1(-m-\alpha;1-\alpha;x_1) L_m^{\alpha}(x)\}.
\end{aligned} \quad (13)$$

Further the auxiliary functions $\vartheta_i(x_1; x)$ ($i = 1,2,3$), which properties are described in [4], introduced into consideration are sought in the form:

$$\begin{aligned}
\vartheta_i(x_1;x) = A_{i1} Y_1(x_1;x) + A_{i2} Y_2(x_1;x) + \\
+ A_{i3} \vartheta_C(x),
\end{aligned} \quad (14)$$

here $\vartheta_C$ is the particular solution of the inhomogeneous equation (6). In our case

$$\vartheta_C(x) = \frac{Cr_0}{D_0(1-\sigma)} e^{-x}\left(1 - \frac{1}{\sigma x}\right). \quad (15)$$

As a result we receive:

$$\begin{aligned}
\vartheta_1(x_1;x) &= [1 - B_1 \vartheta_C(x_1)] Y_1(x_1;x) + \\
&+ \left[\frac{\sigma}{\alpha_0 x_1} + B_1 \vartheta_C'(x_1)\right] Y_2(x_1;x) + B_1 \vartheta_C(x), \\
\vartheta_2(x_1;x) &= -B_2 \vartheta_C(x_1) Y_1(x_1;x) - \\
&- \left[B_2 \vartheta_C'(x_1) - \frac{r_0}{\alpha_0 D(x_1)} x_1^{\frac{1}{\alpha_0}-1}\right] Y_2(x_1;x) + \\
&+ B_2 \vartheta_C(x), \\
\vartheta_3(x_1;x) &= B_3\{-\vartheta_C(x_1) Y_1(x_1;x) - \\
&- \vartheta_C(x_1) Y_2(x_1;x) + \vartheta_C(x)\},
\end{aligned} \quad (16)$$

where

$$B_1 = \frac{1}{\vartheta_C'(x_1)}\left[-Y_1''(x_1;x_1) + \frac{\sigma}{\alpha_0 x_1} Y_2''(x_1;x_1) + \right.$$
$$\left. + \frac{1}{\alpha_0 x_1^2}\left(\sigma + \frac{1}{\alpha_0}\right)\right],$$
$$B_2 = -\frac{r_0}{\alpha_0 D(x_1)\vartheta_C'(x_1)} x_1^{\frac{1}{\alpha_0}-1} \times$$
$$\times\left[Y_1''(x_1;x_1) + \frac{1}{x_1} Y_2''(x_1;x_1) + \frac{1}{D(x_1)}\frac{dD(x_1)}{dx}\right],$$
$$B_3 = \frac{r_0^2}{\alpha_0^2} x_1^{\frac{2}{\alpha_0}-2} \frac{1}{D(x_1)\vartheta_C'(x_1)}.$$

It should be noted that in consideration of the combined plates with the piecewise variable rigidity the Cauchy functions and the auxiliary functions $\vartheta_i$ are different for separate sections. It is valid since the each part has its law of thickness variation $h(x)$ and its own parameters' values. Therefore we introduce the appropriate notation $Y_1^{(1)}$, $Y_2^{(1)}$, $Y_1^{(2)}$, $Y_2^{(2)}$ and $\vartheta_i^{(1)}$, $\vartheta_i^{(2)}$. Let that the combined plate, shown on the Fig.2, is made from the isotropic material that is $n^2 = 1$. We assume that the Poisson's ratio is $\sigma = 1/3$. The plate's thickness in the first section when $0{,}5 \le x \le 1{,}0$ is approximated by the formula (11) when $m = 2$. On the second section $1{,}0 \le x \le 2{,}0$ the plate's thickness is approximated by the same formula (11) when the parameter $m = 1$. The plate's thickness in the place of the sections' joint $x = x_2 = 1{,}0$ is continuous.

We denote as $\vartheta_0$, $M_0$, $Q_0$ correspondingly the angle of rotation, the moment and the force on the inner contour of the plate. The expression for the angles of rotation for the first section $x_1 \le x \le x_2$ is

$$\vartheta = \vartheta_I = \vartheta_0 \vartheta_1^{(1)}(x_0;x) + M_0 r_0 \vartheta_2^{(1)}(x_0;x) + + Q_0 r_0^2 \vartheta_3^{(1)}(x_0;x). \quad (17)$$

For the second section when $x_2 \le x \le x_3$ the angles of rotation are determined by the formula

$$\vartheta = \vartheta_{II} = \vartheta_1(x_1)\vartheta_1^{(2)}(x_1;x) + + M_1(x_1) r_0 \vartheta_2^{(2)}(x_1;x) + Q_1(x_1) r_0^2 \vartheta_3^{(2)}(x_1;x), \quad (18)$$

where $\vartheta_1(x_1)$, $M_1(x_1)$, $Q_1(x_1)$ are received by the use of the formulae (17) and (4).

The expressions for the deflections can be also received. The proposed method can be successfully applied for the combined plates with the piecewise thickness variation consisting of several parts.

## 4. THE CONCLUSION

The work develops the analytical method of the combined plates with the piecewise variable thickness computation. The constructions under study have the circular shape and consist of several parts with different laws of thickness variation. These parts may be made from the same or from the different materials which can be isotropic or orthotropic. In the places of the separate sections joint the thickness can be continuous or discontinuous. For the receiving of the solutions the theory of the special functions is used. The solutions are obtained in closed form and expressed in terms of Lagerr's polynomials and the confluent hypergeometric functions.

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# TRANSVERSE OSCILLATIONS OF THE BEAM ON AN ELASTIC BASE IF THE BOUNDARY CONDITIONS CHANGE

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**Annotation**: The article deals with the proper transverse oscillations of a beam with free edges while the conditions of support on an elastic base change, taking into account its own weight and the influence of the attached mass m1. The problem of determining the forces in the beam is being solved taking into account the dynamic load F(t) applied at an arbitrary point d while the conditions for the support of a part of the beam on an elastic base change.
The conditions that must be taken into account while analyzing the dynamic action of the structure under the influence of variable loads in the case of changes in the conditions of support on an elastic base are formulated.

**Keywords**: ground base, beam on an elastic foundation, the initial parameters method, natural oscillation frequencies, forced oscillations, dynamic analysis.

# ПОПЕРЕЧНЫЕ КОЛЕБАНИЯ БАЛКИ НА УПРУГОМ ОСНОВАНИИ при изменении условий опирания

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**Аннотация**: В работе изучаются собственные поперечные колебания балки со свободными краями при изменении условий опирания на упругое основание с учетом собственного веса и влияния присоединенной массы m1. Решается задача по определению усилий в балке с учетом динамической нагрузки F(t) приложенной в произвольной точке d при изменении условий опирания части балки на упругое основание.
Сформулированы условия, которые необходимо учитывать при анализе динамического поведения конструкции под действием переменных нагрузок в случае изменения условий опирания на упругое основание.

**Ключевые слова**: грунтовое основание, балка на упругом основании, метод начальных параметров, свободные колебания, вынужденные колебания, динамический анализ.

## 1. INTRODUCTION

In order to fulfill the requirements of mechanical safety of buildings and structures, which are regulated by law [1] and have been developed in modern normative and technical documents [2, 3], it is urgent to study structural systems that change the design scheme for various reasons during local destruction [4, 5, 6]. Taking into account the affecting of sudden local destruction on the stress-strain state and dynamics of structures is an urgent need for predicting their work and assessing the bearing capacity and / or stability. Such structural systems include structures lying on the ground, which can be considered in their design as beams on an elastic foundation. To date, there are a number of works [7, 8, 9] devoted to the study of dynamic processes caused by the sudden formation of defects in beams with partial support on an elastic foundation.

## 2. MODELS AND METHODS

We consider a "beam-base" system, in which the beam was initially completely on an elastic foundation, but when a defect suddenly formed under a part of the beam, the base was excluded from power work of this structure (Figure 1). Figure 1 shows that the left side of the beam with length $\alpha L$ is located on an elastic foundation with a constant coefficient $r_0$, the right side of the beam with length $\beta L$ is cantilever. It is of interest to solve the problem of determining the natural frequencies and forms of transverse vibrations of a beam with free edges, in the case of an added mass $m_1$ and a dynamic load $F(t)$ applied at an arbitrary point $d$ when a part of the base under the right part of the beam suddenly has been excluded.

The differential equation of forced transverse vibrations of a beam on an elastic foundation of constant cross-section, taking into account the resistance forces for any law of change of the disturbing force $q(x, t)$, has the form [9–11]:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\mu\frac{\partial^2 y(x,t)}{\partial x^2}+ \\ +2\alpha\frac{\partial^5 y(x,t)}{\partial x^4 \partial t}+r_0 b\, y(x,t)=q(x,t), \tag{1}$$

where $E$ is elasticity modulus of a beam material; $I$ is inertia moment of a beam cross section, $y(x,t)$ is transverse deflection of the beam axis in the section $x$; $q(x,t)$ – disturbing load that changes its value in time $t$; $\mu = q/g$: $q$ – evenly distributed load (dead load) attached along the beam; $g$ – acceleration of gravity; α – coefficient characterizing internal friction of material; $r_0 by(x,t)$ – the intensity of the reaction of the elastic Winkler foundation that varies its values along the length of the beam [10, 11, 12]; $r_0$ – modulus of subgrade reaction; $b$ – width of the beam.

We solved the problem in three stages using the method of initial parameters.

At the first stage, we determined the natural transverse vibrations of the beam taking into account its own weight, and at the second stage – taking into account its own weight and the added mass $m_1$. At the third stage, we solved the problem taking into account the disturbing force, which varies in time according to the harmonic law $F(t) = F \sin yt$ and is applied at an arbitrary point $d$. Here: F is the amplitude value of the disturbing force; γ is the angular frequency of change in the disturbing force.

The first stage.

Let us determine the circular frequencies and forms of natural transverse vibrations of a beam with free edges of length $L$ and flexural rigidity $EI$ (Figure 1).

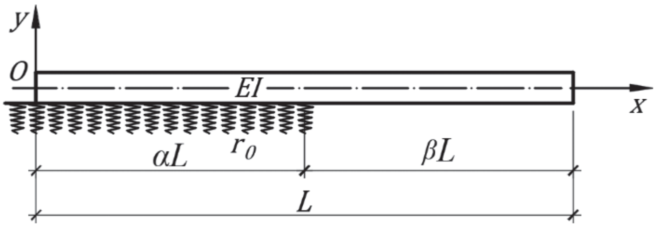



*Figure 1. Beam with free edges, the left part of which αL is located on an elastic foundation.*

It is known [13] that the dissipation of vibration energy on the frequencies and modes of natural vibrations of building structures affects only slightly, attenuation in their calculations is usually neglected.

A simple periodic solution to the equation of natural vibrations of the beam (1) is the main vibration, which changes according to the harmonic law:

$$y(x,t) = \varphi(x)\sin(\omega t + \alpha), \tag{2}$$

where $\varphi(x)$ – function that establishes the distribution law of the maximum deviations of the points of the beam axis from the equilibrium position; $\alpha$ – initial phase of oscillation; $\omega = \omega_\varkappa$ – the circular frequency of natural transverse vibrations of the beam at the base, and $\omega = \omega_k$ – circular frequency of natural transverse vibrations of a beam without a base, (rad / s).

Using the method of separation of variables, problem (2) can be reduced to the equation of natural vibrations for the left side of the beam $\alpha L$ on the basis of:

$$\varphi^{IV}(x) + \varkappa^4\varphi(x) = 0\,, \tag{3}$$

where we accepted designation:

$$\varkappa^4 = \frac{\mu\omega_\varkappa^2 - r}{EI}. \tag{4}$$

For the right side of the beam $\beta L$ without a base, the equation of natural vibrations is:

$$\varphi^{IV}(x) + k^4\varphi(x) = 0\,, \tag{5}$$

where we accepted designation:

$$k^4 = \frac{\mu\omega_k^2}{EI}\,. \tag{6}$$

The solution of equations (3) and (5) is conveniently represented in the form of Krylov functions:

$$\left.\begin{aligned} S(x) &= \tfrac{1}{2}(ch\lambda x + cos\lambda x),\\ T(x) &= \tfrac{1}{2}(sh\lambda x + sin\lambda x),\\ U(x) &= \tfrac{1}{2}(ch\lambda x - cos\lambda x),\\ V(x) &= \tfrac{1}{2}(sh\lambda x - sin\lambda x). \end{aligned}\right\}, \tag{7}$$

where $\lambda = \varkappa$ corresponds to the beam laying on an elastic foundation and $\lambda = k$ corresponds to the beam without foundation.

Let us write down the values of the boundary conditions for a beam with free edges on an elastic foundation:

$$\left.\begin{aligned} x=0: M(0)=Q(0)=0 \\ x=L: M(L)=Q(L)=0 \end{aligned}\right\} \tag{8}$$

For an arbitrary section of the beam in the first section $0 \le x_1 \le \alpha L$, which is located on an elastic foundation, displacements and forces are determined by the equations:

$$\left.\begin{aligned} y_{1i}(x_1) &= y_{10i}\,S(\varkappa_i x_1) + \theta_{10i}\frac{1}{\varkappa_i}T(\varkappa_i x_1) \\ \theta_{1i}(x_1) &= y_{10i}\,\varkappa_i V(\varkappa_i x_1) + \theta_{10i}S(\varkappa_i x_1) \\ M_{1i}(x_1) &= -EJy_{10i}\,\varkappa_i^2 U(\varkappa_i x_1) - EJ\,\theta_{10i}\varkappa_i V(\varkappa_i x_1) \\ Q_{1i}(x_1) &= -EJy_{10i}\,\varkappa_i^3 T(\varkappa_i x_1) - EJ\theta_{10i}\varkappa_i^2 U(\varkappa_i x_1) \end{aligned}\right\} \tag{9}$$

Here *i=1, 2, 3, etc.*

In the second section of the beam without a base $0 \le x_2 \le \beta L$ displacements and forces for an arbitrary section are determined:

$$\left.\begin{aligned} y_{2i}(x_2) &= y_{20i}\,S(k_i x_2) + \frac{\theta_{20i}}{k_i}T(k_i x_2) - \frac{M_{20i}}{k_i^2 EJ}U(k_i x_2) - \frac{Q_{20i}}{k_i^3 EJ}V(k_i x_2) \\ \theta_{2i}(x_2) &= y_{20i}\,k_i V(k_i x_2) + \theta_{20i}S(k_i x_2) - \frac{M_{20i}}{k_i EJ}T(k_i x_2) - \frac{Q_{20i}}{k_i^2 EJ}U(k_i x_2) \\ M_{2i}(x_2) &= -EJy_{20i}\,k_i^2 U(k_i x_2) - EJ\,\theta_{20i}k_i V(k_i x_2) + M_{20i}\,S(k_i x_2) + \frac{Q_{20i}}{k_i}T(k_i x_2) \\ Q_{2i}(x_2) &= -EJy_{20i}\,k_i^3 T(k_i x_2) - EJ\theta_{20i}k_i^2 U(k_i x_2) + M_{20i}\,k_i V(k_i x_2) + Q_{20i}S(k_i x_2) \end{aligned}\right\} \tag{10}$$

Using the conditions of conjugation of the sections $\alpha L$ and $\beta L$, we express the displacements and forces of the second section through the initial parameters of the first section:

$$\left.\begin{aligned} y_{2i}(x_2) &= y_{10i}\left[S(\varkappa_i\,\alpha L)S(k_i x_2) + V(\varkappa_i\alpha L)\frac{\varkappa_i}{k_i}T(k_i x_2) + U(\varkappa_i\alpha L)U(k_i x_2)\frac{\varkappa_i^2}{k_i^2} + T(\varkappa_i\alpha L)V(k_i x_2)\frac{\varkappa_i^3}{k_i^3}\right] + \\ &+\theta_{10i}\left[\frac{1}{\varkappa_i}T(\varkappa_i\,\alpha L)S(k_i x_2) + S(\varkappa_i\alpha L)\frac{1}{k_i}T(k_i x_2) + V(\varkappa_i\alpha L)U(k_i x_2)\frac{\varkappa_i}{k_i^2} + U(\varkappa_i\alpha L)V(k_i x_2)\frac{\varkappa_i^2}{k_i^3}\right] \\ \theta_{2i}(x_2) &= y_{10i}\left[k_i S(\varkappa_i\,\alpha L)\,V(k_i x_2) + \varkappa_i V(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i x_2) + \frac{\varkappa_i^3}{k_i^2}T(\varkappa_i\alpha L)U(k_i x_2)\right] + \\ &+\theta_{10i}\left[\frac{k_i}{\varkappa_i}T(\varkappa_i\,\alpha L)\,V(k_i x_2) + S(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i}{k_i}V(\varkappa_i\alpha L)T(k_i x_2) + U(\varkappa_i\alpha L)\frac{\varkappa_i^2}{k_i^2}U(k_i x_2)\right] \\ M_{2i}(x_2) &= -EJy_{10i}\left[k_i^2 S(\varkappa_i\,\alpha L)U(k_i x_2) + \varkappa_i k_i\,V(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i x_2) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)\,T(k_i x_2)\right] - \\ &-EJ\theta_{10i}\left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i x_2) + k_i S(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i\,V(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i x_2)\right] \\ Q_{2i}(x_2) &= -EJy_{10i}\left[k_i^3 S(\varkappa_i\alpha L)T(k_i x_2) + \varkappa_i k_i^2 V(\varkappa_i\alpha L)\,U(k_i x_2) + \varkappa_i^2 k_i U(\varkappa_i\alpha L)\,V(k_i x_2) + \varkappa_i^3 T(\varkappa_i\alpha L)S(k_i x_2)\right] - \\ &-EJ\theta_{10i}\left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\alpha L)T(k_i x_2) + k_i^2\,S(\varkappa_i\alpha L)\,U(k_i x_2) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i^2 U(\varkappa_i\alpha L)\,S(k_i x_2)\right] \end{aligned}\right\} \tag{11}$$

Using the boundary conditions on the right edge (8) at $x_2 = \beta L$, we obtain the system of equations:

$$\begin{cases} M_{2i}(\beta L) = -EJy_{10i}\left[k_i^2 S(\varkappa_i\,\alpha L)U(k_i\beta L) + \varkappa_i k_i\,V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i\beta L) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)\,T(k_i\beta L)\right] - \\ \quad -EJ\theta_{10i}\left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i\beta L) + k_i S(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i\,V(\varkappa_i\alpha L)\,S(k_i\beta L) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i\beta L)\right] = 0 \\ Q_{2i}(\beta L) = -EJy_{10i}\left[k_i^3 S(\varkappa_i\alpha L)T(k_i\beta L) + \varkappa_i k_i^2 V(\varkappa_i\alpha L)\,U(k_i\beta L) + \varkappa_i^2 k_i U(\varkappa_i\alpha L)\,V(k_i\beta L) + \varkappa_i^3 T(\varkappa_i\alpha L)S(k_i\beta L)\right] - \\ \quad -EJ\theta_{10i}\left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\alpha L)T(k_i\beta L) + k_i^2\,S(\varkappa_i\alpha L)\,U(k_i\beta L) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)\,S(k_i\beta L)\right] = 0 \end{cases} \quad (12)$$

For a nontrivial solution of equations (12), it is necessary that the determinant, composed of the coefficients at arbitrary constants $EJ\,y_{10}$ and $EJ\,\theta_{10i}$, be equal to zero:

$$\begin{aligned} D = &\left[k_i^2 S(\varkappa_i\,\alpha L)U(k_i\beta L) + \varkappa_i k_i\,V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i\beta L) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)\,T(k_i\beta L)\right] * \\ &* \left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\alpha L)T(k_i\beta L) + k_i^2\,S(\varkappa_i\alpha L)\,U(k_i\beta L) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)\,S(k_i\beta L)\right] - \\ &- \left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i\beta L) + k_i S(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i\,V(\varkappa_i\alpha L)\,S(k_i\beta L) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i\beta L)\right] * \\ &* \left[k_i^3 S(\varkappa_i\alpha L)T(k_i\beta L) + \varkappa_i k_i^2 V(\varkappa_i\alpha L)\,U(k_i\beta L) + \varkappa_i^2 k_i U(\varkappa_i\alpha L)\,V(k_i\beta L) + \varkappa_i^3 T(\varkappa_i\alpha L)S(k_i\beta L)\right] = 0 \end{aligned} \quad (13)$$

The roots of equation (13) are the countless row of values $k_i$ and $\varkappa$. In order to solve the equation, we introduce the relation $k_i = \varepsilon_i\,\varkappa_i$. Here $\varepsilon$ is constant value. For each root value $k_i$ and $\varkappa$ a certain angular frequency of natural transverse vibrations corresponds.

Using expression (4), we obtain a formula for determining $\omega_{i\varkappa}$ circular frequencies of natural transverse vibrations of a part of a beam $\alpha L$ on an elastic foundation:

$$\omega_{i\varkappa} = \sqrt{\frac{EI\lambda_{i\varkappa}^4}{\mu(\alpha L)^4} + \frac{r}{\mu}}, \quad (14)$$

where $\lambda_{i\varkappa} = \varkappa_i \alpha L$, and $i = 1, 2, 3$ *etc.* – frequency sequence number.

For a part of the beam $\beta L$ without a base, using (6), we get:

$$\omega_{ik} = \sqrt{\frac{EI\,\lambda_{ik}^4}{\mu(\beta L)^4}}, \quad (15)$$

where $\lambda_{ik} = k_i \beta L$.

Let us determine the natural angular frequencies of transverse vibrations of the beam parts $\alpha L$ on the base and $\beta L$ without the base, which form the spectra $\omega_{1\varkappa} < \omega_{2\varkappa} <\ldots< \omega_{n\varkappa}$ and $\omega_{1k} < \omega_{2k} <\ldots< \omega_{nk}$.

To determine the modes of natural vibrations, we substitute the values of the roots value $k_i$ and $\varkappa_i$ into the solution of the first equation (11), which will determine the values of the relative ordinates *i*-th of that form of natural vibrations.

The second stage.

Let us determine the natural angular frequencies and forms of transverse vibrations, taking into account the own weight and the added mass $m_1$ at point $d$ (Figure 2).

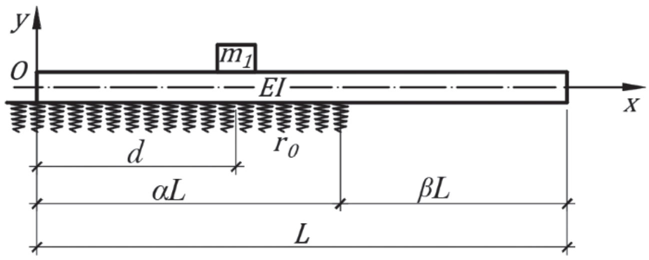



*Figure 2. Beam with added mass $m_1$.*

For an arbitrary section of the beam in the first section $0 \le x_1 \le \alpha L$ displacements and forces are determined by equations (9) only up to the point of application of the mass. For $x_1 > d$ free vibrations of the beam occur with the inertial force *I*. At point *d* we add the inertial force *I* and compose the system of equations:

$$\left.\begin{aligned}
y_{1i}(x_1) &= y_{10i}\,S(\varkappa_i x_1) + \frac{\theta_{10i}}{\varkappa_i}T(\varkappa_i x_1) + \frac{I}{\varkappa_i^3 EI}V(\varkappa_i(x_1-d))\\
\theta_{1i}(x_1) &= y_{10i}\,\varkappa_i V(\varkappa_i x_1) + \theta_{10i}S(\varkappa_i x_1) + \frac{I}{\varkappa_i^2 EI}U(\varkappa_i(x_1-d))\\
M_{1i}(x_1) &= -EJy_{10i}\,\varkappa_i^2 U(\varkappa_i x_1) - EJ\,\theta_{10i}\varkappa_i V(\varkappa_i x_1) - \frac{I}{\varkappa_i}T(\varkappa_i(x_1-d))\\
Q_{1i}(x_1) &= -EJy_{10i}\,\varkappa_i^3 T(\varkappa_i x_1) - EJ\theta_{10i}\varkappa_i^2 U(\varkappa_i x_1) - IS(\varkappa_i(x_1-d))
\end{aligned}\right\}, \qquad (16)$$

where $I = m_1\omega_i^2\left[y_0\,S(\varkappa_i d) + \frac{\theta_0}{\varkappa_i}T(\varkappa_i d)\right]$. (17)

Further, we have composed formulas for determining the deflections, angles of rotation, moments and shear forces of the second section of the beam without a base $0 \le x_2 \le \beta L$ using (9) and the conjugation conditions:

$$\left.\begin{aligned}
y_{2i}(x_2) &= y_{10i}\left[S(\varkappa_i\,\alpha L)S(k_i x_2) + V(\varkappa_i\alpha L)\frac{\varkappa_i}{k_i}T(k_i x_2) + U(\varkappa_i\alpha L)U(k_i x_2)\frac{\varkappa_i^2}{k_i^2} + T(\varkappa_i\alpha L)V(k_i x_2)\frac{\varkappa_i^3}{k_i^3}\right] +\\
&+\theta_{10i}\left[\frac{1}{\varkappa_i}T(\varkappa_i\,\alpha L)S(k_i x_2) + S(\varkappa_i\alpha L)\frac{1}{k_i}T(k_i x_2) + V(\varkappa_i\alpha L)U(k_i x_2)\frac{\varkappa_i}{k_i^2} + U(\varkappa_i\alpha L)V(k_i x_2)\frac{\varkappa_i^2}{k_i^3}\right] +\\
&+\frac{I}{EI}\left[V(\varkappa_i(\alpha L-d))\frac{S(k_i x_2)}{\varkappa_i^3} + U(\varkappa_i(\alpha L-d))\frac{T(k_i x_2)}{\varkappa_i^2 k_i} + T(\varkappa_i(\alpha L-d))\frac{U(k_i x_2)}{\varkappa_i k_i^2} + S(\varkappa_i(\alpha L-d))\frac{V(k_i x_2)}{k_i^3}\right]\\
\theta_{2i}(x_2) &= y_{10i}\left[k_i S(\varkappa_i\,\alpha L)\,V(k_i x_2) + \varkappa_i V(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i x_2) + \frac{\varkappa_i^3}{k_i^2}T(\varkappa_i\alpha L)U(k_i x_2)\right] +\\
&+\theta_{10i}\left[\frac{k_i}{\varkappa_i}T(\varkappa_i\,\alpha L)\,V(k_i x_2) + S(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i}{k_i}V(\varkappa_i\alpha L)T(k_i x_2) + U(\varkappa_i\alpha L)\frac{\varkappa_i^2}{k_i^2}U(k_i x_2)\right] +\\
&+\frac{I}{EI}\left[V(\varkappa_i(\alpha L-d))\frac{k_i V(k_i x_2)}{\varkappa_i^3} + U(\varkappa_i(\alpha L-d))\frac{S(k_i x_2)}{\varkappa_i^2} + T(\varkappa_i(\alpha L-d))\frac{T(k_i x_2)}{\varkappa_i k_i} + S(\varkappa_i(\alpha L-d))\frac{U(k_i x_2)}{k_i^2}\right]\\
M_{2i}(x_2) &= -EJy_{10i}\left[k_i^2 S(\varkappa_i\,\alpha L)U(k_i x_2) + \varkappa_i k_i\,V(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i x_2) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)\,T(k_i x_2)\right] -\\
&-EJ\theta_{10i}\left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i x_2) + k_i S(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i\,V(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i x_2)\right] -\\
&-\frac{I}{\varkappa_i^3}\left[k_i^2 V(\varkappa_i(\alpha L-d))U(k_i x_2) + \varkappa_i k_i U(\varkappa_i(\alpha L-d))V(k_i x_2) + \varkappa_i^2 T(\varkappa_i(\alpha L-d))S(k_i x_2) + \frac{\varkappa_i^3}{k_i}S(\varkappa_i(\alpha L-d))T(k_i x_2)\right]\\
Q_{2i}(x_2) &= -EJy_{10i}[k_i^3 S(\varkappa_i\alpha L)T(k_i x_2) + \varkappa_i k_i^2 V(\varkappa_i\alpha L)\,U(k_i x_2) + \varkappa_i^2 k_i U(\varkappa_i\alpha L)\,V(k_i x_2) + \varkappa_i^3 T(\varkappa_i\alpha L)S(k_i x_2)] -\\
&-EJ\theta_{10i}\left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\alpha L)T(k_i x_2) + k_i^2\,S(\varkappa_i\alpha L)\,U(k_i x_2) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i^2 U(\varkappa_i\alpha L)\,S(k_i x_2)\right] -\\
&-\frac{I}{\varkappa_i^3}[k_i^3 T(k_i x_2)V(\varkappa_i(\alpha L-d)) + \varkappa_i\,k_i^2 U(k_i x_2)U(\varkappa_i(\alpha L-d)) + \varkappa_i^2 k_i V(k_i x_2)T(\varkappa_i(\alpha L-d)) + \varkappa_i^3 S(k_i x_2)S(\varkappa_i(\alpha L-d))]
\end{aligned}\right\} (18)$$

We denote:

$$a_1 = \left[k_i^2 S(\varkappa_i\,\alpha L)U(k_i\beta L) + \varkappa_i k_i\,V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i\beta L) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)\,T(k_i\beta L)\right];$$

$$a_2 = \left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i\beta L) + k_i S(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i\,V(\varkappa_i\alpha L)\,S(k_i\beta L) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)\,T(k_i\beta L)\right];$$

$$a_3 = \Big[k_i^2 V(\varkappa_i(\alpha L-d))U(k_i\beta L) + \varkappa_i k_i U(\varkappa_i(\alpha L-d))V(k_i\beta L) + \varkappa_i^2 T(\varkappa_i(\alpha L-d))S(k_i\beta L) + \\ + \frac{\varkappa_i^3}{k_i}S(\varkappa_i(\alpha L-d))T(k_i\beta L)\Big];$$

$$a_4 = [k_i^3 S(\varkappa_i\alpha L)T(k_i\beta L) + \varkappa_i k_i^2 V(\varkappa_i\alpha L)\,U(k_i\beta L) + \varkappa_i^2 k_i U(\varkappa_i\alpha L)\,V(k_i\beta L) + \varkappa_i^3 T(\varkappa_i\alpha L)S(k_i\beta L)];$$

$$a_5 = \left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\alpha L)T(k_i\beta L) + k_i^2\,S(\varkappa_i\alpha L)\,U(k_i\beta L) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)\,S(k_i\beta L)\right];$$

$$a_6 = [k_i^3 T(k_i\beta L)V(\varkappa_i(\alpha L-d)) + \varkappa_i\,k_i^2 U(k_i\beta L)U(\varkappa_i(\alpha L-d)) + \varkappa_i^2 k_i V(k_i\beta L)T(\varkappa_i(\alpha L-d)) + \\ + \varkappa_i^3 S(k_i\beta L)S(\varkappa_i(\alpha L-d))].$$

Using the boundary conditions on the right edge (8) at $x_2 = \beta L$, taking into account the inertial force $I$ (17), we obtain the system of equations:

$$\begin{cases} y_{10i}\left[a_1 + \frac{m_1\omega_i^2}{\varkappa_i^3 EJ} S(\varkappa_i d)a_3\right] + \theta_{10i}\left[a_2 + \frac{m_1\omega_i^2}{\varkappa_i^4 EJ} T(\varkappa_i d)a_3\right] = 0 \\ y_{10i}\left[a_4 + \frac{m_1\omega_i^2}{\varkappa_i^3 EJ} S(\varkappa_i d)a_6\right] + \theta_{10i}\left[a_5 + \frac{m_1\omega_i^2}{\varkappa_i^4 EJ} T(\varkappa_i d)a_6\right] = 0 \end{cases} \quad (19)$$

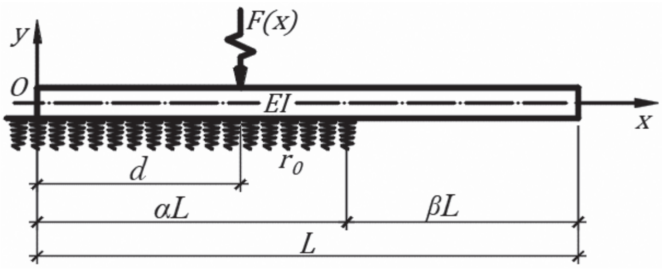
The determinant of this system:

$$D = \left[a_1 + \frac{m_1\varkappa_i}{\mu}\left(1 + \frac{r}{\varkappa_i^4 EJ}\right) S(\varkappa_i d)a_3\right]\left[a_5 + \frac{m_1\varkappa_i}{\mu}\left(1 + \frac{r}{\varkappa_i^4 EJ}\right) T(\varkappa_i d)a_6\right] -$$
$$-\left[a_2 + \frac{m_1\varkappa_i}{\mu}(1 + \frac{r}{\varkappa_i^4 EJ})T(\varkappa_i d)a_3\right]\left[a_4 + \frac{m_1\varkappa_i}{\mu}(1 + \frac{r}{\varkappa_i^4 EJ})S(\varkappa_i d)a_6\right] = 0 \quad (20)$$

Defining a set of values $k_i$ and $\varkappa$ we perform introducing constant $\varepsilon_i$. Using expressions (14) and (15), we determine the values $\omega_{i\varkappa}$ circular frequencies of natural transverse vibrations of a part of the beam $\alpha L$ on an elastic foundation and the values $\omega_{ik}$ for part of the $\beta L$ beam without base.
In order to determine the modes of natural vibrations, the values of the roots $k_i$ and $\varkappa$ substitute in the solution of the first equation (18), which determines the values of the relative ordinates of i-th form of natural vibrations.

<u>The third stage.</u>

Let us determine the efforts under the action of a dynamic load $F(t) = F \sin yt$, applied at an arbitrary point $d$ (Figure 3) for the same beam.




*<u>Figure 3</u>. Beam with dynamic force F(t)*

Let's return to the differential equation of forced vibrations of the beam (1). We assume that the disturbing force acts according to the law $q(x,t) = q(x) \sin yt$. Assuming that forced vibrations also change according to a harmonic law:

$$y(x,t) = \varphi(x)\sin(yt), \quad (21)$$

we obtain an inhomogeneous differential equation of forced vibrations of a beam on an elastic foundation:

$$\varphi^{IV}(x) + \varkappa^4 \varphi(x) = q(x), \quad (22)$$

where: $\varkappa^4 = \frac{\mu\gamma^2 - r}{EI}$. (23)

For a beam without a base, the inhomogeneous differential equation of forced vibrations takes the form:

$$\varphi^{IV}(x) + k^4 \varphi(x) = q(x), \quad (24)$$

where: $k^4 = \frac{\mu\gamma^2}{EI}$. (25)

We have obtain the general solutions of the inhomogeneous equations (22) and (24) as the sum of the general solutions of the homogeneous equation and the particular solution, which depends on the type of load. Further, using the method of initial parameters, we have obtain universal formulas for determining deflections, angles of rotation, moments and shear forces for an arbitrary section of the beam in the general case of the action of a disturbing load $q(x,t)$.
We use the values of the boundary conditions on the left and right edges of the beam (8).
For an arbitrary section of the beam in the first section $0 \le x_1 \le \alpha L$, that located on an elastic foundation and under the action of a dynamic load $F(t)$, that applied at an arbitrary point $d$, displacements and forces are determined by the equations:

$$\left.\begin{aligned}
y_{1i}(x_1) &= y_{10i}\, S(\varkappa_i x_1) + \theta_{10i}\frac{1}{\varkappa_i}T(\varkappa_i x_1) + \frac{F\sin\gamma t}{\varkappa_i{}^3 EI}V[\varkappa_i\,(x_1-d)]\\
\theta_{1i}(x_1) &= y_{10i}\,\varkappa_i V(\varkappa_i x_1) + \theta_{10i}S(\varkappa_i x_1) + \frac{F\sin\gamma t}{\varkappa_i{}^2 EI}U[\varkappa_i\,(x_1-d)]\\
M_{1i}(x_1) &= -EJy_{10i}\,\varkappa_i^2 U(\varkappa_i x_1) - EJ\,\theta_{10i}\varkappa_i V(\varkappa_i x_1) - \frac{F\sin\gamma t}{\varkappa_i}T[\varkappa_i\,(x_1-d)]\\
Q_{1i}(x_1) &= -EJy_{10i}\,\varkappa_i^3 T(\varkappa_i x_1) - EJ\theta_{10i}\varkappa_i^2 U(\varkappa_i x_1) - F\sin\gamma t\;\, S[\varkappa_i\,(x_1-d)]
\end{aligned}\right\}\quad(26)$$

In the second section of the beam without a base $0 \le x_2 \le \beta L$ the displacements and forces for an arbitrary section are determined by (10). Using the conditions of conjugation of the sections and, expressing the displacements and forces of the second section through the initial parameters of the first section, we get:

$$\left.\begin{aligned}
y_{2i}(x_2) &= y_{10i}\left[S(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{\varkappa_i}{k_i}V(\varkappa_i\alpha L)\,T(k_i x_2) + \frac{\varkappa_i^2}{k_i^2}U(\varkappa_i\alpha L)U(k_i x_2) + \frac{\varkappa_i^3}{k_i^3}T(\varkappa_i\alpha L)V(k_i x_2)\right] +\\
&+\theta_{10i}\left[\frac{1}{\varkappa_i}T(\varkappa_i\alpha L)\,S(k_i x_2) + \frac{1}{k_i}S(\varkappa_i\alpha L)\,T(k_i x_2) + \frac{\varkappa_i}{k_i^2}V(\varkappa_i\alpha L)U(k_i x_2) + \frac{\varkappa_i^2}{k_i^3}U(\varkappa_i\alpha L)V(k_i x_2)\right] +\\
&+\frac{F\sin\gamma t}{EI}\left[\frac{V[\varkappa_i(\alpha L-d)]}{\varkappa_i{}^3}S(k_i x_2) + \frac{U[\varkappa_i(\alpha L-d)]}{\varkappa_i{}^2 k_i}T(k_i x_2) + \frac{T[\varkappa_i(\alpha L-d)]}{\varkappa_i k_i^2}U(k_i x_2) + \frac{S[\varkappa_i\,(\alpha L-d)]}{k_i^3}V(k_i x_2)\right]\\
\theta_{2i}(x_2) &= y_{10i}\left[k_i S(\varkappa_i\alpha L)\,V(k_i x_2) + \varkappa_i V(\varkappa_i\alpha L)S(k_i x_2) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)T(k_i x_2) + \frac{\varkappa_i^3}{k_i^2}T(\varkappa_i\alpha L)U(k_i x_2)\right] +\\
&+\theta_{10i}\left[\frac{k_i}{\varkappa_i}T(\varkappa_i\alpha L)\,V(k_i x_2) + S(\varkappa_i\alpha L)S(k_i x_2) + \frac{\varkappa_i}{k_i}V(\varkappa_i\alpha L)T(k_i x_2) + \frac{\varkappa_i^2}{k_i^2}U(\varkappa_i\alpha L)U(k_i x_2)\right] +\\
&+\frac{F\sin\gamma t}{EI}\left[\frac{V[\varkappa_i(\alpha L-d)]}{\varkappa_i{}^3}k_i V(k_i x_2) + \frac{U[\varkappa_i(\alpha L-d)]}{\varkappa_i{}^2}S(k_i x_2) + \frac{T[\varkappa_i(\alpha L-d)]}{\varkappa_i k_i}T(k_i x_2) + \frac{S[\varkappa_i(\alpha L-d)]}{k_i^2}U(k_i x_2)\right]\\
M_{2i}(x_2) &= -EJy_{10i}\left[k_i^2 S(\varkappa_i\alpha L)U(k_i x_2) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i x_2) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)T(k_i x_2)\right] -\\
&-EJ\theta_{10i}\left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i x_2) + k_i S(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i V(\varkappa_i\alpha L)S(k_i x_2) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)T(k_i x_2)\right] -\\
&-\frac{F\sin\gamma t}{\varkappa_i{}^3}\left[k_i^2 U(k_i x_2)V[\varkappa_i(\alpha L-d)] + \varkappa_i\,k_i V(k_i x_2)U[\varkappa_i(\alpha L-d)] + \varkappa_i^2 T[\varkappa_i(\alpha L-d)]S(k_i x_2) + +\frac{\varkappa_i^3}{k_i}S[\varkappa_i(\alpha L-d)]T(k_i x_2)\right]\\
Q_{2i}(x_2) &= -EJy_{10i}[k_i^3 T(k_i x_2)S(\varkappa_i\alpha L) + \varkappa_i k_i^2 U(k_i x_2)V(\varkappa_i\alpha L) + \varkappa_i^2 k_i V(k_i x_2)U(\varkappa_i\alpha L) + \varkappa_i^3 S(k_i x_2)T(\varkappa_i\alpha L)] -\\
&-EJ\theta_{10i}\left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\,\alpha L)T(k_i x_2) + k_i^2 U(k_i x_2)S(\varkappa_i\alpha L) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i x_2) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i x_2)\right] -\\
&-\frac{F\sin\gamma t}{\varkappa_i{}^3}\left[k_i^3 T(k_i x_2)V[\varkappa_i(\alpha L-d)] + \varkappa_i k_i^2 U[\varkappa_i(\alpha L-d)]\,U(k_i x_2) + \varkappa_i^2 k_i V(k_i x_2)T[\varkappa_i(\alpha L-d)] + +\varkappa_i^3 S(k_i x_2)S[\varkappa_i(\alpha L-d)]\right]
\end{aligned}\right\}\quad(27)$$

We denote:

$$a_1 = \left[k_i^2 S(\varkappa_i\alpha L)U(k_i\beta L) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i\beta L) + \frac{\varkappa_i^3}{k_i}T(\varkappa_i\alpha L)T(k_i\beta L)\right];$$

$$a_2 = \left[\frac{k_i^2}{\varkappa_i}T(\varkappa_i\,\alpha L)U(k_i\beta L) + k_i S(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i V(\varkappa_i\alpha L)S(k_i\beta L) + \frac{\varkappa_i^2}{k_i}U(\varkappa_i\alpha L)T(k_i\beta L)\right];$$

$$a_3 = \left[k_i^2 U(k_i\beta L)V[\varkappa_i(\alpha L-d)] + \varkappa_i\,k_i V(k_i\beta L)U[\varkappa_i(\alpha L-d)] + \varkappa_i^2 T[\varkappa_i(\alpha L-d)]S(k_i\beta L) + \right.$$
$$\left. + \frac{\varkappa_i^3}{k_i}S[\varkappa_i(\alpha L-d)]T(k_i\beta L)\right];$$

$$a_4 = [k_i^3 T(k_i\beta L)S(\varkappa_i\alpha L) + \varkappa_i k_i^2 U(k_i\beta L)V(\varkappa_i\alpha L) + \varkappa_i^2 k_i V(k_i\beta L)U(\varkappa_i\alpha L) + \varkappa_i^3 S(k_i\beta L)T(\varkappa_i\alpha L)];$$

$$a_5 = \left[\frac{k_i^3}{\varkappa_i}T(\varkappa_i\,\alpha L)T(k_i\beta L) + k_i^2 U(k_i\beta L)S(\varkappa_i\alpha L) + \varkappa_i k_i V(\varkappa_i\alpha L)V(k_i\beta L) + \varkappa_i^2 U(\varkappa_i\alpha L)S(k_i\beta L)\right];$$

$$a_6 = \left[k_i^3 T(k_i\beta L)V[\varkappa_i(\alpha L-d)] + \varkappa_i k_i^2 U[\varkappa_i(\alpha L-d)]\,U(k_i\beta L) + \varkappa_i^2 k_i V(k_i\beta L)T[\varkappa_i(\alpha L-d)] + \right.$$
$$\left. + \varkappa_i^3 S(k_i\beta L)S[\varkappa_i(\alpha L-d)]\right].$$

Using the boundary conditions on the right edge (8) for $x_2 = \beta L$, we obtain a system of equations for determining $y_{10i}$ and $\theta_{10i}$:

$$\begin{cases} a_1 y_{10i} + a_2 \theta_{10i} = \dfrac{-F \sin \gamma t}{EJ \varkappa_i^3} a_3 \\ a_4 y_{10i} + a_5 \theta_{10i} = \dfrac{-F \sin \gamma t}{EJ \varkappa_i^3} a_6 \end{cases} \quad (28)$$

Using (26) and (28), at a given frequency of forced oscillations $\gamma$, we determine $\varkappa$ and $k$:

$$\varkappa = \sqrt[4]{\frac{\mu\gamma^2 - r}{EJ}}, \quad (29)$$

$$k = \sqrt[4]{\frac{\mu\gamma^2}{EJ}}. \quad (30)$$

Applying equations (26) and (27) taking into account certain values of the roots $\varkappa$ and $k$, that corresponds to given frequency $\gamma$ of forced vibrations, and values of $F(t)$, we have determine forces in the beam under forced vibrations.

## 3. RESULTS AND ANALYSIS

Initial for calculations: beam width $b$=1.25 m, height $h$=1.5 m, length $L$=12.0 m, elasticity modulus of material $E$=2,1x10$^6$ t/m$^2$, modulus of subgrade reaction $r_0$=5000 t/m$^3$, force $F$=10.0 t, mass $m_1$ = $F/g$= 1.0194 t.

At the first and second stages, according to the results of calculations of the beam with $\alpha L = \beta L$, the values of the roots $\varkappa_i$ and $k_i$ of the equations (13) and (20) are adopted such that $\varepsilon_i$ from $k_i = \varepsilon_i \varkappa_i$ equal 0.5; 1.0 and 2.0. Also, the value $\varepsilon_i$ is taken from the condition of equality of natural frequencies of transverse vibrations $\omega_{i\varkappa} = \omega_{ik}$ of two parts of beam. Root values $\varkappa_i$ and $k_i$, indicated in column 7 of tables 1 and 2 for a beam on a full base are defined in [15], in column 8 for a beam without a base – in [14]. The calculation results are presented in table 1.

At the first stage, the first three modes of beam vibrations were constructed with $\alpha L = \beta L$ without added mass $m_1$ (Figures 4, 5 and 6) corresponding to natural frequencies for $\varepsilon$.

Further, at the second stage, according to the results of calculations of a beam with an added mass $m_1$ located in a quarter of the beam $d = L / 4$ at $\alpha L =$

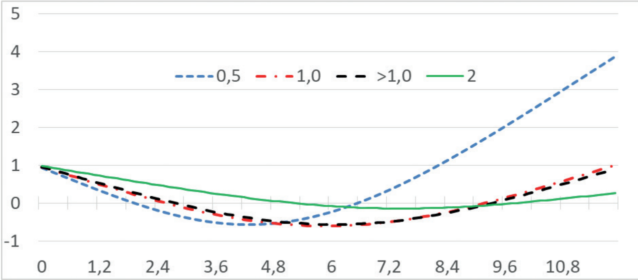



*Figure 4. 1st mode of vibration with $\alpha L = \beta L$ without mass $m_1$*

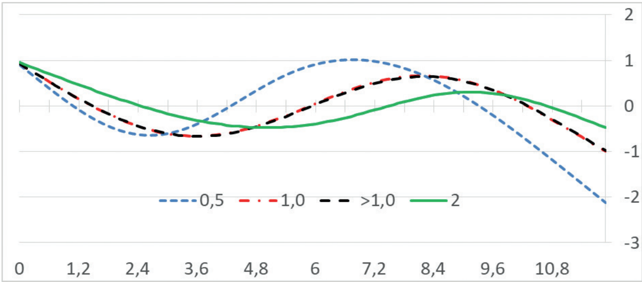



*Figure 5. The 2nd mode of vibration at $\alpha L = \beta L$ without mass $m_1$*

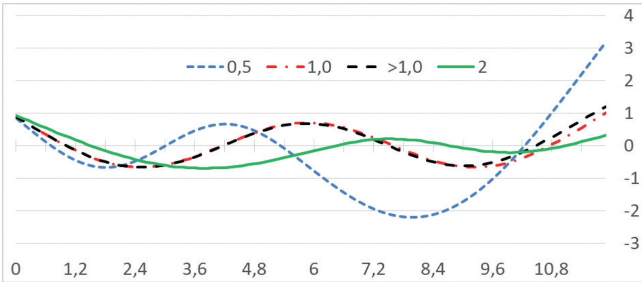



*Figure 6. The 3rd mode of vibration at $\alpha L = \beta L$ without mass $m_1$*

***Table 1. Roots and natural angular frequencies (rad / sec) of transverse vibrations for $\alpha L = \beta$.***

| | | $\varepsilon_i$ | | | | Beam on the base | Beam without base |
|---|---|---|---|---|---|---|---|
| | | 0.5 | 1.0 | $\omega_{i\varkappa} = \omega_{ik}$ | 2.0 | | |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 mode | $\varkappa_1$ | 0.5264 | 0.3942 | 0.3762 | 0.2632 | 0.3942 | - |
| | $k_1$ | 0.2632 | | 0.4108 | 0.5264 | - | 0.3942 |
| | $\omega_1^{\varkappa}$ | 254.7 | 161.7 | 228.9 | 156.2 | 161.7 | - |
| | $\omega_1^{k}$ | 59.4 | 133.3 | | 375.8 | - | 133.3 |
| 2 mode | $\varkappa_1$ | 0.8739 | 0.6544 | 0.6507 | 0.4369 | 0.6544 | - |
| | $k_2$ | 0.4369 | | 0.6582 | 0.8738 | - | 0.6544 |
| | $\omega_2^{\varkappa}$ | 661.4 | 378.5 | 587.6 | 287.4 | 378.5 | - |
| | $\omega_2^{k}$ | 163.7 | 367.3 | | 1035.5 | - | 367.3 |
| 3 mode | $\varkappa_1$ | 1.2566 | 0.9163 | 0.9149 | 0.6283 | 0.9163 | - |
| | $k_3$ | 0.6283 | | 0.9177 | 1.2566 | - | 0.9163 |
| | $\omega_3^{\varkappa}$ | 1357.5 | 726.0 | 1142.2 | 549.7 | 726.0 | - |
| | $\omega_3^{k}$ | 338.6 | 720.2 | | 2141.6 | - | 720.2 |

$\beta L$, we obtain the values of the roots $\varkappa$ and $k_i$. The calculation results are presented in Table 2.

The first three modes of vibrations of the beam are constructed for $\alpha L = \beta L$ with the added mass m1 located at the point $d = L / 4$. Vibration modes corresponding to natural frequencies for $\varepsilon$ are presented in Figures 7, 8 and 9.

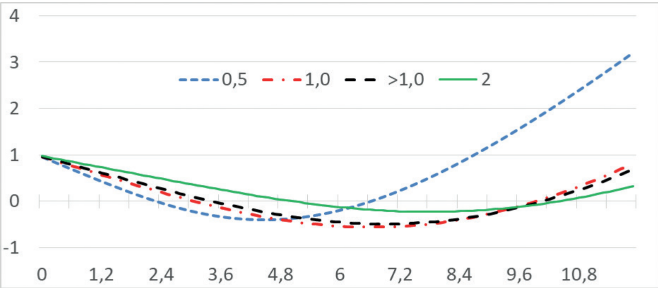



*Figure 7. 1st mode of vibration at $\alpha L = \beta L$ with mass $m_1$ at point $d = L / 4$*

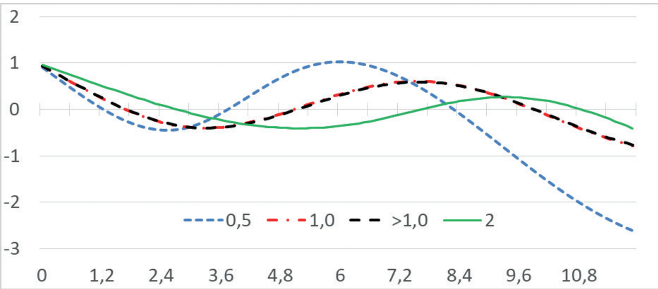



*Figure 8. 2nd mode of vibration at $\alpha L = \beta L$ with mass $m_1$ at point $d = L / 4$*

**Table 2. Roots and natural angular frequencies (rad / s) of transverse vibrations at $\alpha L = \beta$ with mass $m_1$ at point d = L / 4**

| | | $\varepsilon_i$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.5 | 1.0 | $\omega_{i\varkappa} = \omega_{ik}$ | 2.0 | Beam on the base | Beam without base |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 mode | $\varkappa_1$ | 0.4927 | 0.3364 | 0.3034 | 0.2042 | 0.3925 | - |
| | $k_1$ | 0.2463 | | 0.3607 | 0.4085 | - | 0.3942 |
| | $\omega_1^{\varkappa}$, | 352.1 | 197.8 | 176.5 | 137.0 | 160.7 | - |
| | $\omega_1^{k}$ | 82.3 | 153.8 | | 226.3 | - | 133.3 |
| 2 mode | $\varkappa_1$ | 0.8466 | 0.6521 | 0.6477 | 0.4168 | 0.6358 | - |
| | $k_2$ | 0,4233 | | 0.6553 | 0.8336 | - | 0.6544 |
| | $\omega_2^{\varkappa}$ | 980.0 | 590.1 | 582.5 | 266.6 | 358.6 | - |
| | $\omega_2^{k}$ | 243.0 | 576.7 | | 942.4 | - | 367.3 |
| 3 mode | $\varkappa_1$ | 1.3749 | 0.8828 | 0.88 | 0.62 | 0.8936 | - |
| | $k_3$ | 0.6874 | | 0.8831 | 1.24 | - | 0.9163 |
| | $\omega_3^{\varkappa}$ | 2565.7 | 1064.3 | 1057.7 | 536.1 | 691.0 | - |
| | $\omega_3^{k}$ | 640.8 | 1057.0 | | 2085.3 | - | 720.2 |

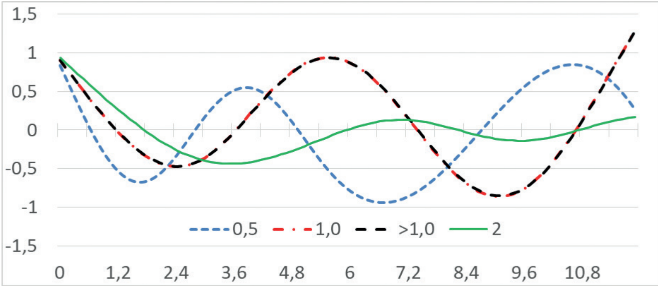



*Figure 9. The 3rd mode of vibration at $\alpha L = \beta L$ with mass $m_1$ at point $d = L / 4$*

At the third stage, an example with the same beam under the action of a disturbing force $F$=10.0 t, applied in the points $d$=$L/2$ and $d$=$L/4$ is considered. Forced vibration frequencies are $\gamma_1$=220 rad/s and $\gamma_2$=400 rad/s. The displacements and forces in the beams are determined at various values $\alpha L$. The figures 10–13 show the bending moment plots.

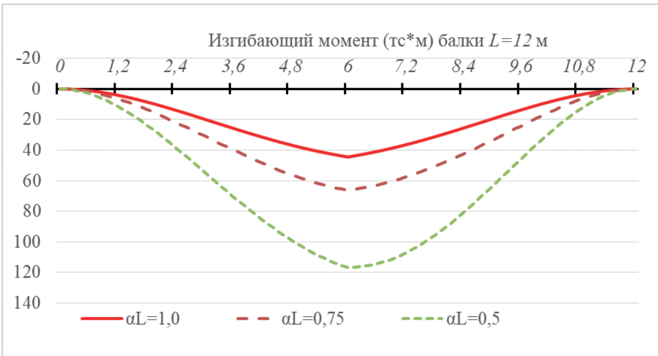



*Figure 10. Diagrams of bending moments under the action of the force F (t) at the point $d = L / 2$ at $\gamma_1$ = 220 rad / s*

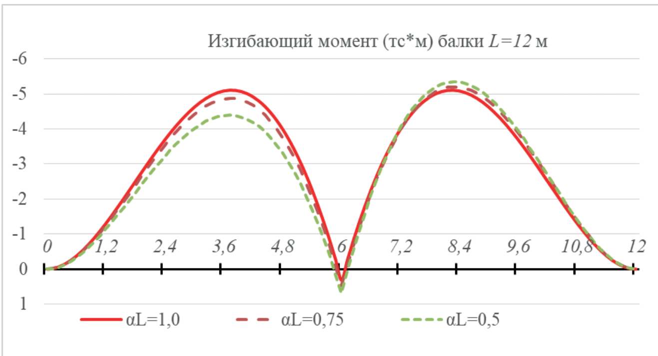



*Figure 11. Diagrams of bending moments under the action of the force F (t) at the point $d = L / 2$ at $\gamma_2$ = 400 rad / s*

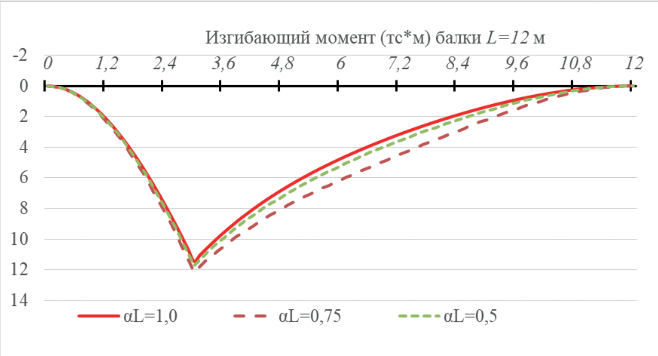



*Figure 12. Diagrams of bending moments under the action of the force F (t) at the point d = L / 4 at* $\gamma_1$ *= 220 rad / s*

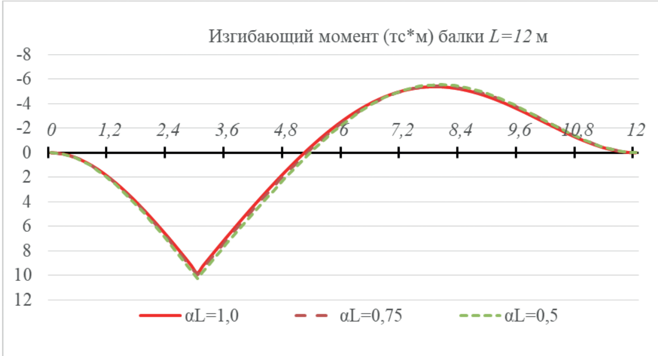



*Figure 13. Diagrams of bending moments under the action of the force F (t) at the point d = L / 4 at* $\gamma_2$ *= 400 rad / s*

Analysis of the calculation results of the third stage allows us to draw the following conclusions.

The action of a disturbing force in the middle of the beam (d = L / 2) with a frequency of forced vibrations $\gamma$ = 220 rad / s, close to the frequency of natural vibrations (for $\omega_{i\varkappa} = \omega_{ik}$) for the first mode of vibration, leads to an increase in displacements and efforts in sections of the beam more than three times when excluding part of the base from the work. Under similar conditions, the action of a disturbing force with a forced vibration frequency $\gamma$ = 400 rad / s, close to the natural vibration frequency for the second form, does not lead to a significant change in the forces in the beam sections when part of the base is excluded from operation.

The action of a disturbing force in a quarter of the beam (d = L/4) with a frequency of forced vibrations $\gamma$ = 220 rad/s and $\gamma$ = 400 rad/s, when part of the base is excluded from the work, does not lead to a significant change in the forces in the beam sections.

## 4. CONCLUSIONS

1) under different conditions of support of the "beam-base" system, different frequencies of natural vibrations based on the results of calculating the roots of the secular equation can correspond to different parts of one beam. The values of the roots that determine the main modes of vibration of the beam as a whole are the values of the roots for a part of the beam on the base, while the natural frequency of the transverse vibrations of the part of the beam on the base is greater than or equal to the natural frequency of vibration of the part of the beam without the base;

2) under different conditions of support of the "beam-base" system, the values of the natural frequencies of transverse vibrations can be equal for different modes of vibration of two different parts of the beam. In this case, the action of a disturbing force with a frequency of forced vibrations equal to the frequency of natural vibrations leads to the formation of resonance for each of the two different modes of vibration of each part of the beam;

3) with the application of an additional mass m1 at the beam point, the vibration frequencies change its values. If the mass of the system increases, then the vibration frequencies of the system decrease and vice versa that corresponds to a similar conclusion for a beam on a full base [15];

4) when performing a dynamic calculation, it is necessary to consider all possible options for the application of masses, taking into account the points of their location in combinations with options for changing the conditions for supporting the beam on an elastic foundation. The number of determined frequencies and modes of natural vibrations for beams on an elastic foundation should not be less than two;

These conditions must be taken into account when analyzing the dynamic behavior of a structure under the action of variable loads in the event of a change in the conditions of bearing on an elastic foundation.

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# THE EXPERIENCE OF THE UNDERGROUND CONSTRUCTION FOR THE COMPLEX OF BUILDINGS ON A SOFT SOIL IN THE CENTER OF ST. PETERSBURG

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**Abstract**: Failures of the important and unique buildings and facilities occur comparatively rarely, but in case of their occurrence, result in significant social and material damage, especially if they are associated with casualties.
The experience of the science technical monitoring of the construction of underground parking in a new hotel in the central part of St. Petersburg is given in the article. The parameters of the main underground structures and problems occurred during their construction are presented. The second part of the paper is devoted to the technologies used during the construction of the second stage of the hotel on the area of the dissembled buildings suffered from serious deformation during the construction of the underground parking for the first stage of the hotel.

**Keywords**: failures of buildings, underground parking, settlements of foundation, excavation pit.

# ОПЫТ ПОДЗЕМНОГО СТРОИТЕЛЬСТВА ДЛЯ КОМПЛЕКСА ЗДАНИЙ НА СЛАБЫХ ГРУНТАХ В ЦЕНТРЕ САНКТ-ПЕТЕРБУРГА

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**Аннотация**: Разрушения ответственных и уникальных зданий и сооружений случаются сравнительно редко, но в случае такого происшествия, результат имеет существенные социальные и материальные последствия, особенно если сопряжены с людскими потерями.
В статье приводится опыт научного сопровождения строительства подземного паркинга в комплексе нового отеля в центральной части Санкт-Петербурга. Приведены параметры основных подземных частей и проблемы, с которыми пришлось столкнуться при строительстве. Вторая часть статьи посвящена технологиям, которые были использованы при строительстве второй стадии отеля на территории разобранных зданий, претерпевших серьезные деформации при строительстве подземного паркинга для первой стадии отеля.

**Ключевые слова**: разрушения зданий, подземный паркинг, осадка фундамента, котлован

## 1. INTRODUCTION

Fortunately, major failures of buildings and facilities occur comparatively rarely, but in case of their occurrence, result in significant social and material damage, especially if they are associated with casualties.
As a rule, the building failures related to ground beds and foundations are the most destructive, and they are caused by errors in designing, construction and operation of facilities. In many cases, such failures result from the integrated interaction of components of such causes.
From the technical point of view, the building failures are due either to soil forced out from underneath the foundation bed (loss of ground bed stability), or to large and unacceptable settlements for given type of a building and their non-uniformity (unacceptable deformation of ground bed). Generally, the destruction of the foundation material is observed not very often.
As a rule, the engineering information on building failures is extremely rare, therefore, all the more

useful to study and analyze the available data in order to accumulate experience and to prevent disasters from occurring, failure conditions or major destructions of buildings in the future.

During the last decades in Saint Petersburg, the cause pattern essentially changed regarding the destruction of adjacent buildings when new buildings are being constructed. Thus, if in 1960–1990 the building deformations were be prevalent during operation (70%) in relation to technological causes of deformations (30%), then since 1990 up to date, this ratio is 35% to 65%.

Departure from the construction practices in new construction and sometimes even simply gross mistakes in construction of the foundation beds and foundations are the causes for major deformations (including hazardous ones) of the load-bearing structures of the buildings and facilities of surrounding development.

## 2. HAZARDOUS DEFORMATIONS OF ADJACENT BUILDINGS IN CONSTRUCTION OF NEVSKY PALACE HOTEL SUBSTRUCTURE ON NEVSKY PROSPEKT, ST.PETERSBURG

In 1992, it was started the reconstruction of the Baltiyskaya Hotel and its remodeling as modern Nevsky Palace Hotel. The high status of the hotel required that an underground car parking was constructed under the main part of the building (Figure 1).

The designing and reconstruction were carried out by foreign companies. It was not planned to underpin the foundations of the hotel facade part facing Nevsky Prospekt and the foundation on the side of the additional entrance from Stremaynnaya Street. These parts of the building are supported by the old rubble stone foundations.

*Figure 1. Diagram of underground car parking construction under Nevsky Palace Hotel*

On the side of Nevsky Prospekt, the foundations under the facade wall are rubble-stone ones formed by bedding limestone on sand-and-lime mortar and have the bed depth of 2,95 m. Timber sleepers were found under their bed.

After the central part of the hotel, a 8-m deep pit was excavated and around it, a diaphragm wall was constructed that was made of secant (having an intersection) augured cast-in-situ piles 20 m long with a cross section of D=0,8 m. The piles were made using the technology of drilling with casing pipe and delivering concrete through a tremie pipe from bottom upwards.

Initially, cracks started appearing in the surrounding buildings during the penetration of the first ten pipes. Apparently, it was occurring the consolidation of the bearing layers of the ground bed – fine-grained and silty sand under the foundation beds of these buildings – and the destructuration of the underlying stratified thixotropic sandy loams and loams. In the process of the construction operations, cracks continued opening and new cracks appeared. It might be associated with the destructurated water-saturated soils flowing in through the open end of the pipe and then being taken off by an auger.

The most considerable damages occurred in the nearby buildings in the process of constructing "even" piles when drilling in the concrete of the previously constructed piles. Obviously, the vibration action that takes place in drilling of the "primary" piles with special-purpose drilling tools provided with three-cone bits around the perimeter resulted in the thixotropic destructuration of soil and the deterioration of its strength and deformation properties. The soil transformed into the running state and, in the absence of so-called "soil plug", easily got through to the bottom of the borehole, which led to an additional scope of the soil excavation in drilling the boreholes and to the development of wider subsidence trough. These deformations has major effect on the further damage to the masonry strength of the nearby buildings. The settlements of 17 and 13 cm occurred at the nearest points of the foundations of the buildings in Nevsky Prospekt that were located nearby the excavation pit.

*a)*

*b)*

*c)*

*Figure 2. General view of hazardousdamages of buildings in the vicinity of hotel in Nevsky Prospekt (a and b) and Stremyannaya Street (c)*

The resulting deformations in the envelopes of the surrounding buildings lead to the relocation of the inhabitants of five buildings in Nevsky Prospekt and neighbor Stremyannaya Street (Fig. 2).

## 3. ATTACHING OF TWO NEW BUILDINGS ON DRILLED CAST-IN-SITU PILES TO EXISTING BUILDING OF OPERATING NEVSKY PALACE HOTEL

At the end of 2005, in the place of the demolished buildings at 55 and 59, Nevsky Prospekt, the works started to construct the foundations of new buildings for the Corinthia Nevsky Palace Hotel.
It was planned to construct Buildings No. 59 (without basement) and No. 55 with a basement 4,5 m deep on drilled cast-in-situ piles 32 m long and 880 and 620 mm in diameter using a casing pipe.
The demolished buildings were seriously damaged in 1992 when a diaphragm wall was constructed by the

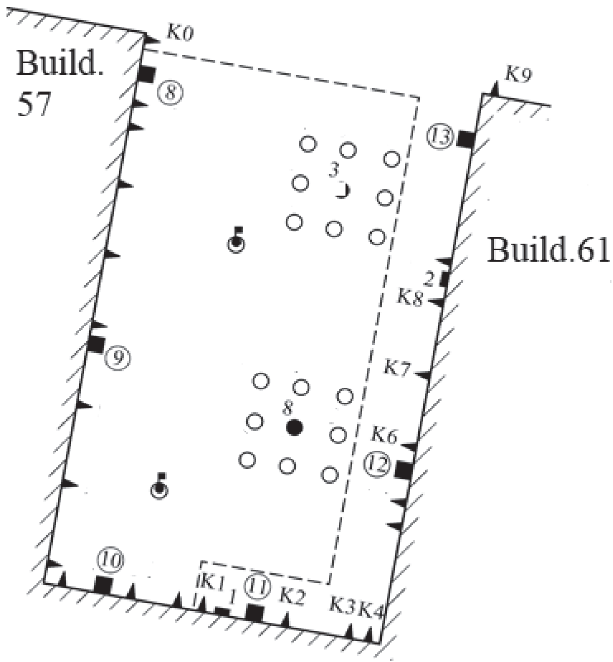



*Figure 3. Layout of seismic pickups, settlement benchmarks and tell-tales;*
*Legend: ■ – seismic receiver installation points;*
*----- – Outline of excavation pit;*
*▬ – Tell-tale and its number;*
*▲ – Settlement benchmarks; Monitoring well o control underground water level (measurement of piezometric level);*
*● – Test piles and its number;*
*○ – Anchor piles*

tangent pile method for the underground car parking. Prior to starting the works, the buildings in Nevsky Prospekt and Stremyannaya Street surrounding the two construction sites were surveyed and fitted with tell-tales installed on the existing cracks and with deformation benchmarks for geodetic monitoring. As an example, Figure 3 shows the as-built diagram of the layout of seismic pickups, settlement benchmarks and tell-tales of the monitoring wells to monitor the underground water levels, etc.

In total, observations were carried out twice a week for 95 benchmarks and more than 50 tell-tales. Four wells were used to monitor the underground water levels [ 1 ].

When carrying out the works related to underpinning of the foundations and ground beds of the buildings surrounding the construction site by the method of injecting cement grout into the contact area, as well as when constructing the foundations of the drilled cast-in-situ piles, their vibration impact on the enclosing structures of the neighboring buildings was controlled.

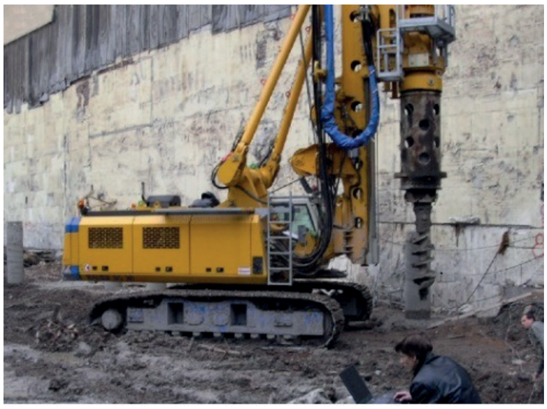
The foundation of the neighboring building, which was erected in the beginning the XIX century, were the stone masonry, but the down part of it put granite steening. This was required use injection of the compound cement mortar to provide the continuity of foundation before piling work (Fig. 4a). Figure 4b shows a process of underpinning the building foundations using the Hilty equipment.

Figure 5a shows a picture of the process of measuring vibrations of the walls of the surrounding buildings. The taken measurements of the vibration acceleration in the load-bearing structures of the buildings allowed establishing, in particular, that the process of underpinning the foundations and ground beds is safe, according to the technology applied, for the walls of the building, and that it was not acceptable to use more than one drilling rig at a time on the construction site. At the simultaneous operation of two and more self-propelled drilling rigs of the BG 25 type, the measured vibration acceleration in the building walls

*a)*

*b)*

*Figure 4. Underpinning of foundations and ground bed of building in Stremyannaya Street (a and b)*



*a)*

*b)*

*Figure 5. The complex science investigations on the construction site: taking vibration measurements during the construction of drilled cast-in-situ piles (a); the stamp test of the soil on the bottom pile's level (b).*

exceeded the maximum permissible values b) and might cause the structures to be destroyed.

Prior to starting the mass construction of the drilled cast-in-situ piles, the static tests were carried out on test piles, and the tests showed that their load-carrying capacity was at least 2000 kN, which is considerably higher than their design load (Figure 6).

Further in the mass construction of the above piles, random sampling was performed regarding the quality and integrity of the body of the drilled cast-in-situ piles by non-destructive testing integrity of the body of the drilled cast-in-situ piles by non-destructive testing methods using seismic-acoustic instrument IDS-1 [ 2 ]

The monitoring of the settlement benchmarks on the neighboring buildings showed that, when

*Figure 6. Static tests of piles using hydraulic jacks*

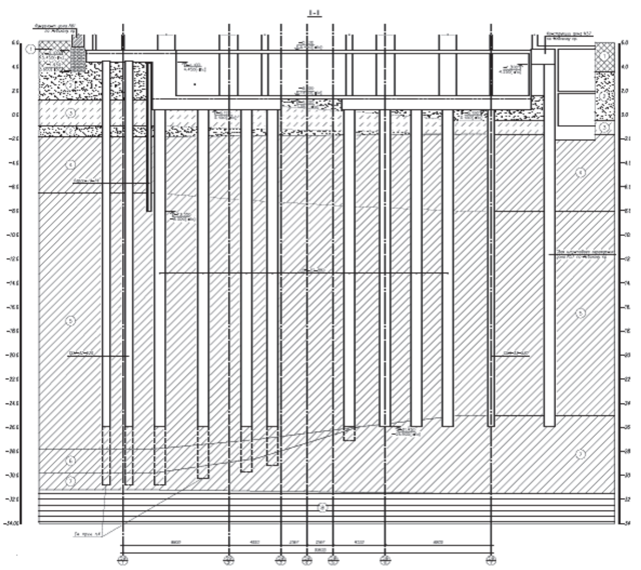


*Figure 7. The cross section of the basement and piles with the difference depth according of the level bearing capacity soils.*

constructing the piles for the new Hotel building at 59 Nevsky Prospekt, the additional deformations of their foundations were less than 20 mm and no damages occurred in their superstructures.

As sheet piling for an excavation pit 4,5 to 6,2 m deep for an underground floor of a new Hotel Building at 55 Nevsky Prospekt, it was used the Larsen IV pile sheeting driven by a Muller non-resonant vibration generator in the area along Stremyannaya Street and the ALCELOR jacked pile sheeting driven by a press system of cassette type installed on a base of a Banut 655 pile-driving machine (Figure 8 a and b).

Back at the time when the non-standard additional settlement started developing, a decision was made to underpin their ground beds and, for one of the wings, to strengthen the superstructures with metal bands.

The pile sheeting was jacked along Building No. 53, Nevsky Prospekt. At different depths along the line of the pile sheeting jacking, various inclusions were encountered in the form of timber sleepers, old rubble stone foundations and large boulders. To withdraw the inclusions, a trench was excavated down to a depth of 2,7 m. This resulted in additional settlement of approximately 25 mm for the buildings at a distance of less than 2 m from the excavation pit. In the following pit excavation, these settlements increased and reached up to 60 mm for individual benchmarks.

The geodetic monitoring of the facade verticality for the existing Hotel building showed its deviation up to 50 mm from the vertical line towards Nevsky Prospekt. In view of that, a prompt decision was made to underpin the foundation of that wall with drilled injected piles 14 m long and 150 mm in diameter. The action taken made it possible to complete the construction of the substructure and to start constructing the superstructures [3].

Further geodetic monitoring of the settlements for the new buildings and the neighboring buildings identified no hazardous tendencies. By the end of the construction, the settlements of the new buildings did not exceed 30 mm, and the settlements of the neighboring buildings stabilized.

At the end of May 2009, the new Hotel buildings were successfully commissioned (Figure 9).

## 4.CONCLUSION

The experience of this construction showed how important is to comply with the requirements of


a)


b)

*Figure 9. New buildings of Corinthia Nevsky Palace Hotel: a – Building No. 59; b – Building No. 55*

the construction operations method and to take into consideration the specific engineering and geological conditions of a given construction site. Our experience shows that even the use of the most state-of-the-art foreign technologies without adapting them to the application in soft water-saturated silty-clayed soils of Saint Petersburg may result in dramatic consequences. Thus, the use of secant augered cast-in-place piles for the pit sheeting without a special-purpose cutting working head that allows minimizing the dynamic action on soft soils becomes unacceptable and hazardous when constructing in the compact building systems. Later on, domestic geotechnical companies began using, in construction of pit sheeting of augered cast-in-place piles, a system of adjoining piles injecting cement mortar between them.

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# OVERVIEW OF THE UNATED STATES AND THE EUROPEAN UNION STANDARDS IN TERMS OF ANALYSIS OF BUILDINGS AND STRUCTURES UNDER SEISMIC WAVE ACTION

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**Abstract**: The article discusses the terms of the US and EU standards (ASCE -7-10, ASCE-4-98, FEMA P-1051/2016, EN 1998-6: 2005) concerning the calculations of earthquake -resistant buildings and structures taking into account wave seismic effects in the ground base. For the considered standards, wave propagation models and accepted approaches to seismic analysis were investigated; limitations on the use of the standard methods were identified.

**Keywords**: Seismic waves, wave model of seismic ground motion, seismic analysis, rotational seismic ground motion, linear dynamic analysis, rotational response spectra.

# АНАЛИЗ НОРМ США И ЕВРОСОЮЗА В ЧАСТИ РАСЧЕТОВ ЗДАНИЙ И СООРУЖЕНИЙ НА ВОЛНОВЫЕ СЕЙСМИЧЕСКИЕ ВОЗДЕЙСТВИЯ

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**Аннотация**: В статье приведен анализ положений ряда сейсмических норм США и Евросоюза (ASCE-7-10, ASCE-4-98, FEMA P-1051/2016, EN 1998-6:2005) по проектированию сейсмостойких зданий и сооружений с учетом волновых сейсмических эффектов в грунтовом основании. Исследованы заложенные в нормы модели распространения волн и принятые подходы к проектному расчету, выявлены ограничения по применению нормативных методик.

**Ключевые слова**: Сейсмические волны, волновая модель сейсмического движения грунта, расчет на сейсмостойкость, ротационное сейсмическое движение грунта, линейный динамический анализ, ротационные спектры ответа.

The wave seismic effects on buildings and structures occur when seismic waves pass through the ground base. Since seismic waves velocities are finite, there is a time-delay between kinematic parameters (displacements, velocities, accelerations) at various points of the ground. For correct analysis of spatial buildings and structures, it is necessary to consider a space-time field of displacements, velocities and accelerations at points of their ground base. As presented in [1–3], the effect of seismic wave propagation is introduced into the analysis by seismic impact vector consisting of three translational and three rotational (angular) components at each point of the ground base. In particular, in [1] is discussed the conditions under which the field of ground wave motions at the base is reduced to a single seismic impact vector applied to the geometric center of the base. Ideas about the rotational components of seismic motion, which must be considered in structural analyses together with translational ones, appear in many scientific publications, see, for example, [4–7]. The need to take into account the rotational seismic motion at the base for some types of buildings and structures is present in foreign standards. This problem has been most fully resolved in the EU building codes, and to a much lesser extent – in the United States.

In the ASCE-7-10 [8], the rotational motion is simulated by random eccentricities for overlaps of structure (the corresponding explanations are given in [9]). A similar approach to accounting for wave phenomena in the engineering design is observed in the American atomic standards ASCE 4-98 ([10], C.3.3.1.2). ASCE 4-98 accepts the hypothesis about vertical propagation of body seismic waves.

The seismic analysis is performed on vertical displacements of the base from P-waves and horizontal displacements from shear waves. Apparent velocity of vertical shear waves on the surface tends to infinity and there are no rotations. The simplified model of seismic motions as vertically propagating body waves should be used with the simultaneous setting of overlap's random eccentricities, for guarantee that the building or structure will not be affected by any unaccounted wave effects. Further in C.3.3.1.2, it is noted the complexity of the real wave motions in the base and the corresponding features of the dynamic behavior of structures, such as associated horizontal, vertical, torsional and rocking motions, depending on the soil parameters, the foundation, the frequency range, etc.

Consider in detail the approach implemented in the European seismic standards EN 1998-6: 2005 [11]. In EN 1998-6:2005, spatial translational and rotational ground motions should be taken into account for tall structures (towers, masts, chimneys, etc.). In 3.1 "Definition of the seismic input" EN 1998-6:2005 it is written: "In addition to the translational components of the earthquake motion, defined in EN 1998-1:2004, 3.2.2 and 3.2.3, the rotational component of the ground motion should be taken into account for tall structures in regions of high seismicity." A Note 1 to p.3.1 states that conditions under which the rotational component of the ground motion should be taken into account in a country, will be found in National Annex. The recommended conditions are structures taller than 80 m in regions where the product $a_g S$ exceeds 0.25$g$, where $a_g$ is the design ground acceleration for type $A$ ground; $S$ is the soil factor; $a_g S$ – design acceleration of soil for a given soil. Informative Annex gives a possible method to define the rotational components of the ground motion and provides guidance for taking them into account in the analysis. It should be noted that the National Annexes of the EU countries (for example, Cyprus, Greece) use Appendix A in its original form without changes [13-14]. An analysis according to the informative Annex A of EN 1998-6: 2005 "Linear dynamic analysis accounting for the rotational components of the ground motion" should be carried out if there are no results of a special study or well-documented field measurements. In these cases, the rotational response spectra may be determined as:

$$R_x^\theta(T) = 1,7\pi S_e(T)/v_s T\ , \quad (1)$$

$$R_y^\theta(T) = 1,7\pi S_e(T)/v_s T\ , \quad (2)$$

$$R_z^\theta(T) = 2,0\pi S_e(T)/v_s T\ , \quad (3)$$

where $R_x^\theta(T)$, $R_y^\theta(T)$, $R_z^\theta(T)$ are the rotation response spectra around $x$, $y$ and $z$ axes, rad/s$^2$; $S_e(T)$ is the elastic response spectra for the horizontal components on the site, $m/s^2$; $T$ is the period, s; $v_s$ is the average S-wave velocity of the top 30 m of the ground profile, m/s.

The velocity $v_s$ is directly evaluated by field measurements, or through the laboratory measurement of the shear modulus $G$ and the soil density $\rho$ as $v_s = \sqrt{G/\rho}$, or $v_s$ is accepted for standard ground type A, B, C and D equal to 800, 580, 270 and 150 m/s, respectively. Rotational response spectra have the same physical meaning as response spectra for translational motion, but in terms of angular accelerations: this is the maximum angular acceleration of an oscillator with natural period $T$ and $a$ damping coefficient $\xi$ in response to ground rotations with peak angular acceleration $\ddot{\theta}$. The analysis is performed simultaneously for three translational and three rotational components of the seismic ground motions.

Appendix A shows the equations of motion for a flat cantilever model (Fig.1), which is described by horizontal translational displacements $u_t$ of the concentrated masses $m_t$ relative to the base. The seismic action is determined as translational horizontal $\ddot{X}$ and rotational $\ddot{\theta}$ ground motions with the corresponding spectra $S_e(T)$ and $R^\theta(T)$. In EN 1998-6:2005, the equations of motion are written as:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = -(\{m\}\ddot{X} + \{mh\}\ddot{\theta}), \quad (4)$$

where $[M]$ = diag$[m_i]$ is diagonal inertia matrix, $[K]$ is the stiffness matrix, $[C]$ is the damping – matrix, $\{m\}$ is vector comprising masses $m_i$, $\{mh\}$ is vector comprising products (Fig.1).

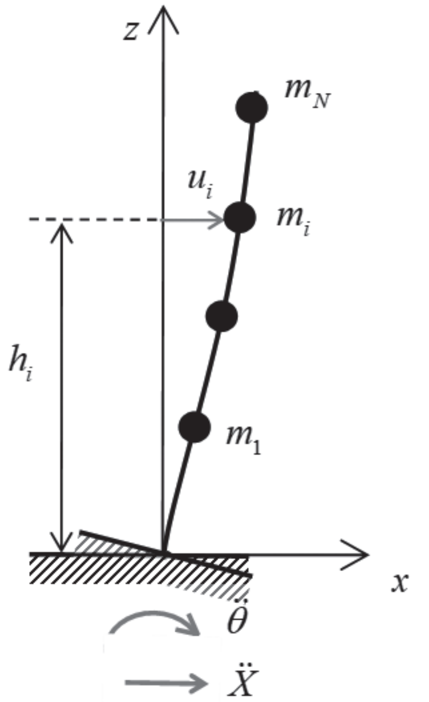


*Figure 1. The flat cantilever mode*

The forces on the right part of (4) are represented as two independent loads. The participation factors are determined for each load. For modal analysis, the participation coefficients of mode k are equal, respectively for the first and second loads in the right part (4):

$$a_{ku} = \frac{\{\Phi^T\}\{m\}}{\{\Phi^T\}[M]\{\Phi\}},\ a_{k\theta} = \frac{\{(\Phi h)^T\}\{m\}}{\{\Phi^T\}[M]\{\Phi\}},$$

where $\{\Phi\}$ is the k-th modal vector; $\{\Phi h\}$ is the vector of the products of the modal amplitude $\Phi$ at the i-th degree of freedom and its elevation $h_i$.

For linear systems in the time domain, full dynamic response to both loads is calculated as superposition of responses for each load. For linear response spectrum method, the resulting dynamic response are found by the rule SRSS (Square Root of the Sum of Squares).

We try to determine the generalized wave model [1–3, 15] corresponding the spectra (1)–(3). In the generalized wave model, it is assumed that translational motion $X_i$ along the i-th axis is caused by shear displacements from SH- and SV-waves and longitudinal displacements from P-waves (Fig. 2):

$$X_1 = u_1 + v_1 + w_1,\ X_2 = u_2 + v_2 + w_2,$$
$$X_3 = u_3 + v_3 + w_3.$$

Without longitudinal displacements from P-waves which do not cause rotations:

$$X_1 = v_1 + w_1, X_2 = u_2 + w_2, X_3 = u_3 + v_3. \tag{5}$$

Further, we assume that all components of the wave motion in (5) are harmonic waves from the Fourier spectrum with the same frequency, wave number, and their own phase delay:

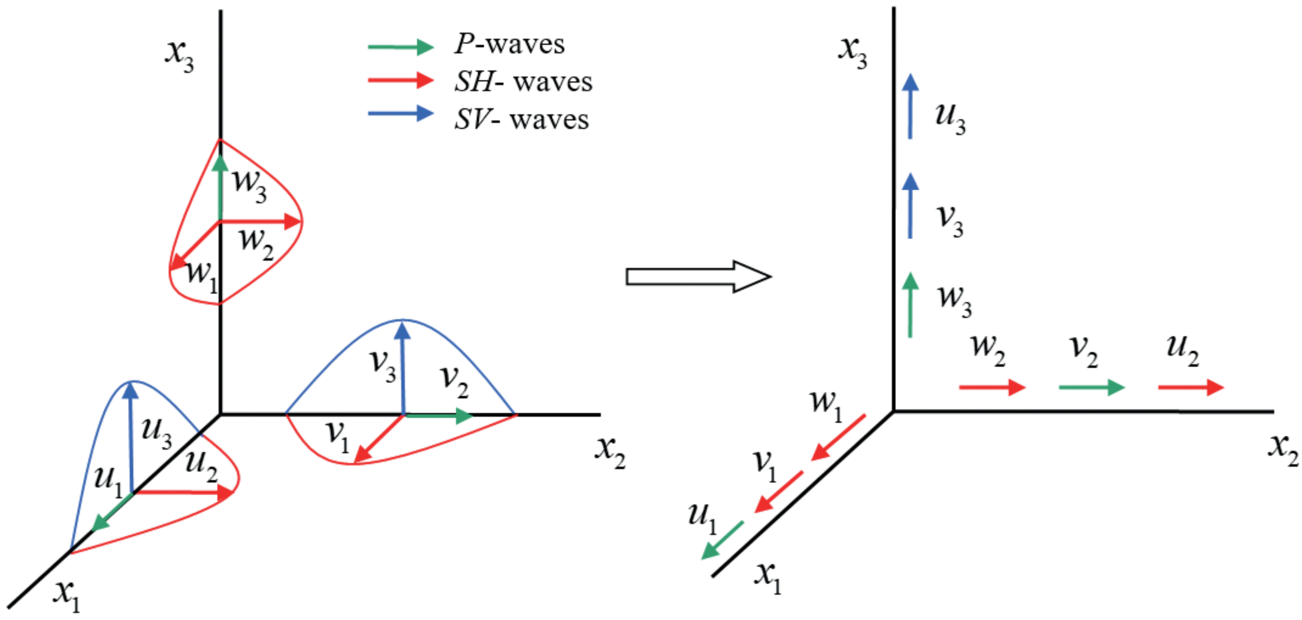


*Figure 2. The generalized wave model*

$$X_1 = v_1 + w_1 =$$
$$= A_{11}\cos(kx_2 + \omega t + \varphi_{11}) + A_{12}\cos(kx_3 + \omega t + \varphi_{12}),$$
$$X_2 = u_2 + w_2 =$$
$$= A_{21}\cos(kx_1 + \omega t + \varphi_{21}) + A_{22}\cos(kx_3 + \omega t + \varphi_{22}),$$
$$X_3 = u_3 + v_3 =$$
$$= A_{31}\cos(kx_1 + \omega t + \varphi_{31}) + A_{32}\cos(kx_2 + \omega t + \varphi_{32}).$$

Accelerations of the translational motion are equal:

$$\ddot{X}_1 = -\omega^2 A_{11}\cos(kx_2 + \omega t + \varphi_{11}) -$$
$$-\omega^2 A_{12}\cos(kx_3 + \omega t + \varphi_{12}),$$
$$\ddot{X}_2 = -\omega^2 A_{21}\cos(kx_1 + \omega t + \varphi_{21}) -$$
$$-\omega^2 A_{22}\cos(kx_3 + \omega t + \varphi_{22}),$$
$$\ddot{X}_3 = -\omega^2 A_{31}\cos(kx_1 + \omega t + \varphi_{31}) -$$
$$-\omega^2 A_{32}\cos(kx_2 + \omega t + \varphi_{32})$$

with maximum absolute values:

$$\max|\ddot{X}_1| = \omega^2 (A_{11} + A_{12}),$$
$$\max|\ddot{X}_2| = \omega^2 (A_{21} + A_{22}),$$
$$\max|\ddot{X}_3| = \omega^2 (A_{31} + A_{32}).$$

Rotational accelerations are calculated using well-known formulas (see, for example, in [1, 2]):

$$\ddot{\theta}_1 = \frac{1}{2}\left(\frac{\partial \ddot{X}_3}{\partial x_2} - \frac{\partial \ddot{X}_2}{\partial x_3}\right) =$$
$$= \omega^2 \frac{k}{2}(A_{32}\sin(kx_2 + \omega t + \varphi_{32}) -$$
$$-A_{22}\sin(kx_3 + \omega t + \varphi_{22})),$$
$$\ddot{\theta}_2 = \frac{1}{2}\left(\frac{\partial \ddot{X}_1}{\partial x_3} - \frac{\partial \ddot{X}_3}{\partial x_1}\right) =$$
$$= \omega^2 \frac{k}{2}(A_{12}\sin(kx_3 + \omega t + \varphi_{12}) -,$$
$$-A_{31}\sin(kx_1 + \omega t + \varphi_{31})),$$
$$\ddot{\theta}_3 = \frac{1}{2}\left(\frac{\partial \ddot{X}_2}{\partial x_1} - \frac{\partial \ddot{X}_1}{\partial x_2}\right) =$$
$$= \omega^2 \frac{k}{2}(A_{21}\sin(kx_1 + \omega t + \varphi_{21}),$$
$$-A_{11}\sin(kx_2 + \omega t + \varphi_{11})).$$

The maximum absolute values of rotational accelerations are equal to

$$\max|\ddot{\theta}_1| = \omega^2 \frac{k}{2}(A_{32} + A_{22}),$$
$$\max|\ddot{\theta}_2| = \omega^2 \frac{k}{2}(A_{12} + A_{31}),$$
$$\max|\ddot{\theta}_3| = \omega^2 \frac{k}{2}(A_{21} + A_{11}).$$

Rotational spectra (1)–(3) are expressed only in terms of acceleration of horizontal translational motion, so $\ddot{X}_3 = 0$ and $A_{31} = A_{32} = 0$, therefore

$$\begin{aligned} &\max|\ddot{X}_1| = \omega^2 (A_{11} + A_{12}) \\ &\max|\ddot{X}_2| = \omega^2 (A_{21} + A_{22}), \\ &\max|\ddot{\theta}_1| = \omega^2 \frac{k}{2} A_{22},\ \max|\ddot{\theta}_2| = \omega^2 \frac{k}{2} A_{12}, \\ &\max|\ddot{\theta}_3| = \omega^2 \frac{k}{2}(A_{21} + A_{11}). \end{aligned} \tag{7}$$

Assuming that the amplitudes in the above formulas are of the same order, we estimate translational and rotational accelerations

$$\frac{\max|\ddot{\theta}_i|}{\max|\ddot{X}_i|} \sim k\ . \tag{8}$$

The estimation (8) shows the ratio of the maximum amplitudes of the rotational and translational components of the seismic impact. For the linear system, the estimation (8) is also true for the translational and rotational response spectra. The wave number $k$ is related to the wavelength $\lambda = v_s T$, and, accordingly, to its period $T$ and phase velocity $v_s$:

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{v_s T}. \tag{9}$$

The spectra (1)-(3) with accounting (8) and (9):

$$\begin{aligned} &R_x^{\theta}(T) = 0{,}85\frac{2\pi}{v_s T} S_e(T), \\ &R_y^{\theta}(T) = 0{,}85\frac{2\pi}{v_s T} S_e(T), \\ &R_z^{\theta}(T) = \frac{2\pi}{v_s T} S_e(T). \end{aligned} \tag{10}$$

Consider in (10) the Type 1 elastic response spectra for horizontal translational motion $S_e(T)$ determined in Table 1 [11, 12].

*Table 1. Type 1 Elastic response spectra*

| Period | Response spectra |
|---|---|
| $0 < T \le T_B$ | $S_e(T) = a_g S\left[1+\frac{T}{T_B}(2,5\eta-1)\right]$ |
| $T_B \le T \le T_C$ | $S_e(T) = a_g S\eta \cdot 2,5$ |
| $T_C \le T \le T_D$ | $S_e(T) = a_g S\eta \cdot 2,5\left[\frac{T_C}{T}\right]$ |
| $T_D \le T \le 4$ c | $S_e(T) = a_g S\eta \cdot 2,5\left[\frac{T_C T_D}{T^2}\right]$ |

Fig. 3 and 4 show graphs of the rotational response spectra (10) and translational response spectra given in Table.1. The translational spectra are shown as a solid line, the rotational spectra as a dotted line. Fig. 3 is drawn for soil A with $V_s$ = 800 m/s, Fig. 4 – for soil *D* with $V_s$ = 150 m/s.

The graphs of the rotational spectra in Fig. 3 and 4 show that the rotational motion corresponding to (1)-(3) is a high-frequency component of the seismic action, the contribution of which to the structural response increases for soft, loose soils. The reduction coefficients in (1)-(3) equal to 0.85 for rotational spectra with respect to two horizontal axes. It seems to have been introduced artificially (for example, to account for the non-synphase of seismic waves or the weakening of the dynamic response due to the scattering of seismic waves).

## CONCLUSIONS

1. The US standards ASCE-7-10 and ASCE-4-98 accepted a model of vertical body wave propagation. In this case, the horizontal and vertical displacements of the base are caused by shear waves and compression waves respectively; there are no rotational components since the apparent velocity of vertical shear waves tends to infinity. The accidental eccentricity is used to indirectly account for various effects, including: plan distributions of mass that differ from those assumed in design, variations in the mechanical properties of structural components, non-uniform yielding of the lateral system, and torsional and rotational ground motions [9]. However, the accidental eccentricity approach cannot be called successful for simulating torsional and rotational ground motions, since the motion of a dynamical system with eccentricities and with ground rotations has different causes and is described by different equations. Simple illustrative examples of the equations of motion can be found in [16].

2. In the European Union standard EN 1998-6:2005 it is proposed a method of analysis of tall structures (towers, masts, chimneys, etc.) for simple flat cantilever (Fig.1) with the equation of motion (4). Rotational response spectra (1)-(3) are expressed through the response spectra of horizontal translational motion. The method is based on a simplified wave model as a composition of SH-waves propagating

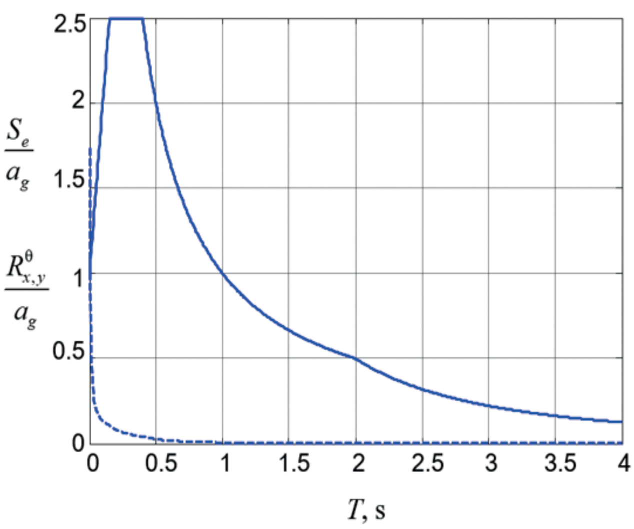


*Figure 3. Translational and rotational response spectra. Ground A, Type I*

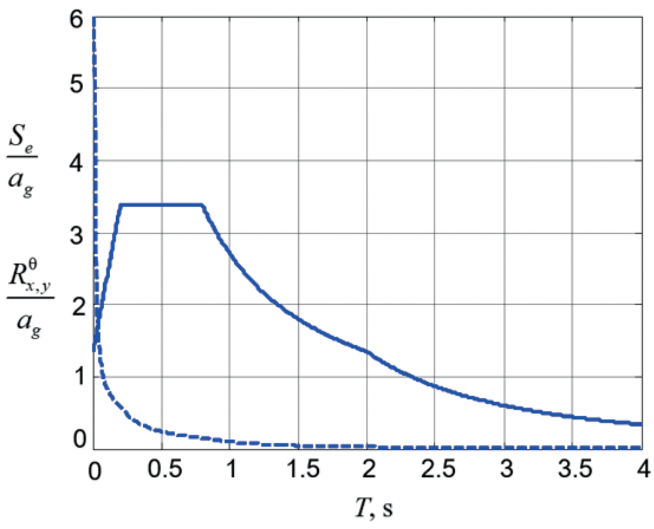


*Figure 4. Translational and rotational response spectra. Ground D, Type I*

in two orthogonal horizontal directions with a finite phase velocity. For this wave model, the rotational response spectra are obtained, and the rules for calculating the resulting forces under the combined action of translational and rotational components of seismic motion are described. The spatial extended and large-span buildings and structures are not considered in the Eurocode. The reason is probably in a lack of scientific and methodological basis.

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# CLASSIFICATION OF INTERNAL RESONANCES IN NONLINEAR FRACTIONALY DAMPED UFLYAND-MINDLIN PLATES

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**Abstract**: In the present paper, the nonlinear free vibrations of fractionally damped plates are studied, equations of motion of which take the rotary inertia and shear deformations into account and involve five coupled nonlinear differential equations in terms of three mutually orthogonal displacements and two angles of rotation. The procedure resulting in decoupling linear parts of equations has been proposed with further utilization of the generalized method of multiple time scales for solving nonlinear governing equations of motion, in so doing the amplitude functions have been expanded into power series in terms of the small parameter and depend on different time scales. The occurrence of the internal or combinational resonances in Uflyand-Mindlin plates has been revealed and classified.

**Keywords**: Nonlinear elastic Uflyand-Mindlin plate, fractional damping, fractional derivative Kelvin-Voigt model, generalized method of multiple time scales

# КЛАССИФИКАЦИЯ ВНУТРЕННИХ РЕЗОНАНСОВ В НЕЛИНЕЙНЫХ ПЛАСТИНКАХ УФЛЯНДА-МИНДЛИНА С ДРОБНЫМ ДЕМПФИРОВАНИЕМ

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**Аннотация**: В данной работе изучаются нелинейные колебания пластинок на основе моделирования сил внешнего демпфирования с помощь производных дробного порядка. При этом используется система пяти нелиненйых уравнений движения, учитывающая деформации сдвига и силы инерции, относиельно трех перемещений  в трех взаимно ортогональных направлениях и двух углов поворота. В  качестве метода решения используется обобщенный метод многих временных масштабов. Выявлены возможные типы внутренних и комбинационных резонансов, которые могут возникать в платинках Уфлянда-Миндлина, и дана их классификация.

**Ключевые слова**: нелинейно упругая пластинка Уфлянда-Миндлина, демпфирование с помощью дробной производной, модель Кельвина-Фойгта с дробной производной, обобщенный метод многих временных масштабов

## 1. INTRODUCTION

Recently the interest to nonlinear dynamic response of viscoelastic plates or elastic plates vibrating in a viscoelastic surrounding medium has been greatly renewed due to the appearance of advanced materials exhibiting nonlinear behavior, and a comprehensive review in the field, including experimental results, could be found in [1–7]. In so doing the damping forces are usually taken into account according to the Rayleigh's hypothesis [2,8], resulting in the modal damping [9],  i.e. it is assumed that each natural mode of vibrations possesses its own damping coefficient dependent on its natural frequency. For describing the viscoelastic features of plates, the Kelvin-Voigt model [5] or standard linear solid model [6] are of frequent use in engineering practice considering either linear or nonlinear springs in viscoelastic elements [10].
The analysis of free undamped [11] and damped [5] vibrations of nonlinear systems is of great importance

for defining the dynamic system's characteristics dependent on the amplitude-phase relationships and modes of vibration. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in strong coupling between the modes of vibrations involved [11–16] and hence in the energy exchange between the interacting modes.

The internal resonance could be observed in the case of some combination of natural frequencies of one and the same type of vibrations. Thus, nonlinear vibrations of rectangular plates, dynamic behavior of which is described by von Karman equations in terms of the plate's deflection and stress function, have been considered in [13] by reducing the governing equations to a set of two modal equations applying the Galerkin procedure.The case of the one-to-one internal resonance (when frequencies of two modes of flexural vibration are equal to each other) accompanied by the external resonance (when the frequency of the harmonic force is close to one of the natural frequency) has been studied.

The one-to-one internal resonance has been investigated also in [14] and [15] for nonlinear vertical vibrations of rectangular plates under the action of harmonic forces acting in the plate's plane [14] and out of the plate's plane [14,15], in so doing a set of three equations in terms of two in-plane displacements and deflection and a set of five equations considering the shear deformations have been used in [14] and [15], respectively. However, considering the inertia forces only for vertical vibrations and utilizing the Galerkin procedure, in both papers a set of two nonlinear equations has been obtained in terms of two flexural modes, which are assumed to be coupled via the one-to-one internal resonance. For the first two natural modes of flexural vibrations, the cases of the 1:2 and 1:3 internal resonances have been also studied in [15].

Another type of the internal resonance has been investigated by Rossikhin and Shitikova [16–20], when one frequency of in-plane vibrations is equal (the 1:1 internal resonance [18,20]) or two times larger (the 1:2 internal resonance [16,19]) than a certain frequency of out-of-plane vibrations. As this takes place, a set of three nonlinear differential equations in terms of three mutually orthogonal displacements has been used considering inertia of all types of vibrations, what allows the authors to study the combinational resonances of the additive and difference types as well [17, 20–22]. Combinational types of the internal resonance result in the energy exchange between three or more subsystems. It should be noted that investigations in this direction were initiated by Witt and Gorelik [23], who pioneered in the theoretical and experimental analysis of the energy transfer from one subsystem to another using the simplest two-degree-of-freedom mechanical system, as an example.

Moreover, in order to study nonlinear free damped vibrations of a thin plate, the viscoelastic Kelvin-Voigt model involving fractional derivative [24] has been utilized, since this model possesses the advantage over the conventional Kelvin-Voigt model [11–15], because it provides the results matching the experimental data. Thus, for example, experimental data on ambient vibrations study for the Vincent-Thomas [25] and Golden Gate [26] suspension bridges have shown that different modes of vibrations possess different magnitudes of damping coefficients. Besides, the increase in the natural frequency results in the decrease in the damping ratio. In order to lead the theoretical investigation in the agreement with the experiment, in 1998 it was suggested in [27] to utilize the fractional derivatives to describe the processes of internal friction occurring in suspension combined systems, what allowed the authors in a natural way to obtain the damping ratios, which depend on natural frequencies.

Nowadays fractional calculus is widely used for solving linear and nonlinear dynamic problems of structural mechanics, what is evident from numerous studies in the field, the overview of which could be found in the state-of-the-art articles by Rossikhin and Shitikova [28,29], wherein the examples of adopting the fractional derivative Kelvin-Voigt, Maxwell and standard linear solid models are provided for single-mass oscillators, rods, beams, plates, and shells.

In particular, linear vibrations of Kirchhoff-Love plates with Kelvin-Voigt fractional damping were considered for rectangular and circular plates, respectively, in [30] and [31] using one equation for vertical vibrations, while utilizing three equations of in-plane and transverse vibrations in

[8,32], and later multiplate systems were analyzed in [28,33]. It has been proved [29,34] that if viscoelastic properties of plates are described by the Kelvin-Voigt model assuming the Poisson’s ratio as the time-independent value (though for real viscoelastic materials the Poisson's ratio is always a time-dependent function [35]), then this case coincides with the case of the dynamic behavior of elastic bodies in a viscoelastic medium. Thus, the authors of [30,31], and not only them, replaced one problem with another, namely: a problem of the dynamic response of viscoelastic Kirchhoff-Love plates in a conventional medium with a problem of dynamic response of elastic Kirchhoff-Love plates in a viscoelastic medium, damping features of which are governed by the fractional derivative Kelvin-Voigt model. The vibration suppression of fractionally damped thin rectangular simply supported plates subjected to a concentrated harmonic loading has been studied recently in [36] in order to minimize the plate deflection at the natural frequencies of the plate, in so doing the vibration suppression is accomplished by attaching multiple absorbers modelled as Kelvin-Voigt fractional oscillators, i.e. generalizing the approach suggested in [28,33].

As for the analysis of nonlinear vibrations of plates, then except the above mentioned papers [16,18–21], the fractional derivative Kelvin-Voigt model was used in [37–42] and fractional derivative standard linear solid model in [7,43,44] but without considering the phenomena of the internal resonance.

Thus, free and forced vertical vibrations of an orthotropic plate have been studied in [37] considering first four modes of flexural vibrations, and during the analysis of force driven vibrations the frequency of a harmonic force was assumed to be equal to one of natural frequencies. The von Karman plate equation with fractional derivative damping was utilized in [38] for analyzing the cases of primary, subharmonic and superharmonic resonance conditions, when the harmonic force frequency, respectively, is approximately equal, three times less or larger than the first or second natural frequency of vertical vibrations. Nonlinear random vibrations of the same plate was studied in [41]. Dynamic nonlinear response to random excitation of a simply supported rectangular plate based on a foundation, damping features of which are described by the fractional derivative Kelvin-Voigt model, has been considered in [40]. The analysis of chaotic vibrations of simply supported nonlinear viscoelastic plate with fractional derivative Kelvin-Voigt model has been carried out in [42] for the case when the plate is subjected to an in-plane harmonic force in one direction and a transverse harmonic force. The Galerkin decomposition has been used to obtain the modal equation of the system, in so doing the authors restricted themselves only by the first mode. The fractional derivative standard linear solid model has been utilized in [44] for a viscoelastic layer for active damping of geometrically nonlinear vibrations of smart composite plates using the higher order plate theory and finite element method with discretizing the plate by eight-node isoparametric quadrilateral elements.

Recently the approaches suggested in [19,20] for solving the problem on free nonlinear vibrations of elastic plates in a viscoelastic medium, damping features of which are governed by the Riemann-Liouville derivatives of the fractional order, and in [45] for studying the dynamic response of the fractional Duffing oscillator subjected to harmonic loading have been generalized for the case of forced vibrations of a simply-supported nonlinear thin elastic plate under the conditions of different internal resonances, when two or three natural modes corresponding to mutually orthogonal displacements are coupled [46–49].

In the present paper, the procedure proposed in [20] for solving the problem of free nonlinear vibrations of elastic plates in a fractional derivative viscoelastic medium, when the damped motion is described by a set of three nonlinear equations, has been extended for the case of free vibrations of a simply-supported fractionally damped nonlinear thin elastic plate, the motion of which is described by five equations involving shear deformations and rotary inertia.

## 2. PROBLEM FORMULATION

In order to consider free damped vibrations of a nonlinear simply-supported rectangular plate, first we recall the equations of motion of a nonlinear elastic rectangular plate, which take into account shear deformations and rotary inertia [50]

$$\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}+$$
$$+\frac{1+\mu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{1-\mu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial y^2}=\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 u}{\partial t^2}, \quad (1)$$

$$\frac{\partial^2 v}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}+$$
$$+\frac{1+\mu}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}+\frac{1-\mu}{2}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x^2}=\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 v}{\partial t^2}, \quad (2)$$

$$k^2\frac{1-\mu}{2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial \psi_x}{\partial x}+\frac{\partial \psi_y}{\partial y}\right)+$$
$$+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\left[\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right]+\frac{1-\mu}{2}\frac{\partial w}{\partial y}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\right)+ \quad (3)$$
$$+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\left[\mu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]+\frac{1-\mu}{2}\frac{\partial w}{\partial x}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\right)=$$
$$=\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 w}{\partial t^2},$$

$$\frac{\partial^2 \psi_x}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 \psi_x}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 \psi_y}{\partial x\partial y}-$$
$$-6k^2\frac{1-\mu}{h^2}\left(\frac{\partial w}{\partial x}+\psi_x\right)=\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 \psi_x}{\partial t^2}, \quad (4)$$

$$\frac{\partial^2 \psi_y}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 \psi_y}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 \psi_x}{\partial x\partial y}-$$
$$-6k^2\frac{1-\mu}{h^2}\left(\frac{\partial w}{\partial y}+\psi_y\right)=\frac{\rho\left(1-\mu^2\right)}{E}\frac{\partial^2 \psi_y}{\partial t^2}, \quad (5)$$

subjected to the initial

$$u|_{t=0}=v|_{t=0}=w|_{t=0}=0,$$
$$\dot{u}|_{t=0}=\dot{v}|_{t=0}=\dot{w}|_{t=0}=0,$$
$$\psi_x\big|_{t=0}=\dot{\psi}_x\big|_{t=0}=0, \quad (6)$$
$$\psi_y\big|_{t=0}=\dot{\psi}_y\big|_{t=0}=0,$$

as well as the boundary conditions (a) along the y-axis direction

$$w\big|_{x=0}=w\big|_{x=a}=0, \qquad u_{x=0}=u\big|_{x=a}=0,$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=\left.\frac{\partial u}{\partial x}\right|_{x=a}=0,$$
$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a}=0, \quad (7)$$
$$\left.\frac{\partial^2 \psi_x}{\partial x^2}\right|_{x=0}=\left.\frac{\partial^2 \psi_x}{\partial x^2}\right|_{x=a}=0,$$

and (b) along the x-axis direction

$$w\big|_{y=0}=w\big|_{y=b}=0, \qquad v\big|_{y=0}=v\big|_{y=b}=0,$$
$$\left.\frac{\partial v}{\partial y}\right|_{y=0}=\left.\frac{\partial v}{\partial y}\right|_{y=b}=0,$$
$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0}=\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=b}=0, \quad (8)$$
$$\left.\frac{\partial^2 \psi_y}{\partial y^2}\right|_{y=0}=\left.\frac{\partial^2 \psi_y}{\partial y^2}\right|_{y=b}=0,$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ and $w = w(x, y, t)$ are the displacements of points located in the plate's middle surface in the $x$-, $y$-, and $z$-directions, respectively, $\psi_x(x, y, t)$ and $\psi_y(x, y, t)$ are the angles of rotation of the normal to the middle surface and in the plane tangent to the lines $z$ and $x$, $k$ is the shear coefficient, $\mu$ is the Poisson's ratio, $a$ and $b$ are the plate's dimensions along the $x$- and $y$-axes, respectively, $h$ is its thickness, and $t$ is the time.

Let us rewrite equations (1)-(8) in the dimensionless form introducing the following dimensionless values:

$$u^*=\frac{u}{a}, \quad v^*=\frac{v}{a}, \quad w^*=\frac{w}{a},$$
$$x^*=\frac{x}{a}, \quad y^*=\frac{y}{b}, \quad (9)$$
$$t^*=\frac{t}{a}\sqrt{\frac{E}{(1-\mu^2)\rho}}.$$

Substituting then (9) in (1)-(8), omitting asterisks for ease of presentation, and introducing the forces of resistance of the surrounding medium, resulting in damped vibrations, as it was suggested in [16,18], yield

$$\begin{aligned}
&u_{,xx}+\frac{1-\mu}{2}\beta_1^2u_{,yy}+\frac{1+\mu}{2}\beta_1v_{,xy}+\\
&+w_{,x}\left(w_{,xx}+\frac{1-\mu}{2}\beta_1^2w_{,yy}\right)+\\
&+\frac{1+\mu}{2}\beta_1^2w_{,y}\,w_{,xy}=\ddot{u}+\chi_1D^\gamma u,
\end{aligned}\tag{10}$$

$$\begin{aligned}
&\beta_1^2v_{,yy}+\frac{1-\mu}{2}v_{,xx}+\frac{1+\mu}{2}\beta_1u_{,xy}+\\
&+\beta_1w_{,y}\left(\beta_1^2w_{,yy}+\frac{1-\mu}{2}w_{,xx}\right)+\\
&+\frac{1+\mu}{2}\beta_1w_{,x}\,w_{,xy}=\ddot{v}+\chi_2D^\gamma v,
\end{aligned}\tag{11}$$

$$\begin{aligned}
&k^2\frac{1-\mu}{2}\left(w_{,xx}+\beta_1^2w_{,yy}+\psi_{x,x}+\beta_1\psi_{y,y}\right)+\\
&+w_{,xx}\left(u_{,x}+\mu\beta_1v_{,y}\right)+\\
&+\beta_1^2w_{,yy}\left(\mu u_{,x}+\beta_1v_{,y}\right)+(1-\mu)\beta_1w_{,xy}\left(\beta_1u_{,y}+\mu v_{,x}\right)\\
&+w_{,x}\left(u_{,xx}+\frac{1-\mu}{2}\beta_1^2u_{,yy}+\frac{1+\mu}{2}\beta_1v_{,xy}\right)\\
&+\beta_1w_{,y}\left(\frac{1-\mu}{2}v_{,xx}+\beta_1^2v_{,yy}+\frac{1+\mu}{2}\beta_1u_{,xy}\right)=\\
&=\ddot{w}+\chi_3D^\gamma w,
\end{aligned}\tag{12}$$

$$\begin{aligned}
&\psi_{x,xx}+\frac{1-\mu}{2}\beta_1^2\psi_{x,yy}+\frac{1+\mu}{2}\beta_1\psi_{y,xy}-\\
&-6k^2\frac{1-\mu}{\beta_2^2}\left(w_{,x}+\psi_x\right)=\ddot{\psi}_x+\chi_4D^\gamma\psi_x,
\end{aligned}\tag{13}$$

$$\begin{aligned}
&\psi_{y,yy}+\frac{1-\mu}{2}\psi_{y,xx}+\frac{1+\mu}{2}\beta_1\psi_{x,xy}-\\
&-6k^2\frac{1-\mu}{\beta_2^2}\left(\beta_1w_{,y}+\psi_y\right)=\ddot{\psi}_y+\chi_5D^\gamma\psi_y,
\end{aligned}\tag{14}$$

where $\beta_1=a/b$ and $\beta_2=h/a$ are the parameters defining the dimensions of the plate, $\chi_i$ $(i=1,2,...,5)$ are damping coefficients, overdots denote time-derivatives, lower indices after a comma label the derivatives with respect to the corresponding coordinates, and $D^\gamma$ is the Riemann-Liouville fractional derivative [51] defined as

$$D^\gamma F=\frac{\partial}{\partial t}\int_0^t\frac{F\left(t-t'\right)dt'}{\Gamma(1-\gamma)t'^\gamma}.\tag{15}$$

## 3. METHOD OG SOLUTION

Let us seek the solution of equations (10)–(14) in the form of expansions in terms of eigen modes of vibration

$$\begin{aligned}
u(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{1mn}(t)\eta_{1mn}(x,y),\\
v(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{2mn}(t)\eta_{2mn}(x,y),\\
w(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{3mn}(t)\eta_{3mn}(x,y),\\
\psi_x(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{4mn}(t)\eta_{4mn}(x,y),\\
\psi_y(x,y,t)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{5mn}(t)\eta_{5mn}(x,y),
\end{aligned}\tag{16}$$

where $x_{imn}(t)$ $(i=1,2,...,5)$ are the generalized displacements corresponding to the plate's in-plane displacements, its deflection and angels of rotation, while the eigen forms satisfying the boundary conditions (7)-(8) have the form

$$\begin{aligned}
&\eta_{1mn}(x,y)=\eta_{4mn}(x,y)=\cos\pi mx\sin\pi ny,\\
&\eta_{2mn}(x,y)=\sin\pi mx\cos\pi ny,\\
&\eta_{3mn}(x,y)=\eta_{5mn}(x,y)=\sin\pi mx\sin\pi ny.
\end{aligned}\tag{17}$$

Substituting (16) and (17) in equations (10)-(14), multiplying then (10)-(14) by $n_{imn}(x,y)$, respectively, integrating over $x$ and $y$, and applying the condition of orthogonality of the eigen modes within the domains $0\le x,y\le 1$, we are led to a set of coupled nonlinear second-order differential equations in $x_{imn}(t)$

$$\ddot{x}_{1mn}+\chi_1D^\gamma x_{1mn}+x_{1mn}S_{11}^{mn}+x_{2mn}S_{12}^{mn}=-F_{1mn},\tag{18}$$

$$\ddot{x}_{2mn}+\chi_2D^\gamma x_{2mn}+x_{1mn}S_{21}^{mn}+x_{2mn}S_{22}^{mn}=-F_{2mn},\tag{19}$$

$$\ddot{x}_{3mn} + \chi_3 D^\gamma x_{3mn} + x_{3mn} s_{33}^{mn} + x_{4mn} s_{34}^{mn} + x_{5mn} s_{35}^{mn} = -F_{3mn} \quad (20)$$

$$\ddot{x}_{4mn} + \chi_4 D^\gamma x_{4mn} + x_{3mn} s_{43}^{mn} + x_{4mn} s_{44}^{mn} + x_{5mn} s_{45}^{mn} = 0, \quad (21)$$

$$\ddot{x}_{5mn} + \chi_5 D^\gamma x_{5mn} + x_{3mn} s_{53}^{mn} + x_{4mn} s_{54}^{mn} + x_{5mn} s_{55}^{mn} = 0, \quad (22)$$

where

$$S_{11}^{mn} = \pi^2 \left( m^2 + \frac{1-\mu}{2} \beta_1 n^2 \right),$$

$$S_{12}^{mn} = S_{21}^{mn} = \pi^2 \frac{1+\mu}{2} \beta_1 mn, \quad (23)$$

$$S_{22}^{mn} = \pi^2 \left( \beta_1 n^2 + \frac{1-\mu}{2} m^2 \right),$$

$$s_{33}^{mn} = k^2 \frac{1-\mu}{2} \pi^2 \left( m^2 + \beta_1 n^2 \right),$$

$$s_{34}^{mn} = k^2 \frac{1-\mu}{2} \pi m,$$

$$s_{35}^{mn} = k^2 \frac{1-\mu}{2} \pi \beta_1 n,$$

$$s_{43}^{mn} = 6k^2 \frac{1-\mu}{\beta_2^2} \pi m,$$

$$s_{53}^{mn} = 6k^2 \frac{1-\mu}{\beta_2^2} \pi \beta_1 n, \quad (24)$$

$$s_{44}^{mn} = \pi^2 \left( m^2 + \frac{1-\mu}{2} \beta_1 n^2 \right) + 6k^2 \frac{1-\mu}{\beta_2^2},$$

$$s_{45}^{mn} = s_{54}^{mn} = \pi^2 \beta_1 \frac{1+\mu}{2} mn,$$

$$s_{55}^{mn} = \pi^2 \left( \beta_1 n^2 + \frac{1-\mu}{2} m^2 \right) + 6k^2 \frac{1-\mu}{\beta_2^2}.$$

Nonlinear parts of equations (18)-(20) have the form

$$F_{1mn} = \sum_{m_1} \sum_{n_1} \sum_{m_2} \sum_{n_2} x_{3m_1 n_1} x_{3m_2 n_2} A_{mn}^{m_1 n_1 m_2 n_2},$$

$$F_{2mn} = \sum_{m_1} \sum_{n_1} \sum_{m_2} \sum_{n_2} x_{3m_1 n_1} x_{3m_2 n_2} B_{mn}^{m_1 n_1 m_2 n_2},$$

$$F_{3mn} = \sum_{m_1} \sum_{n_1} \sum_{m_2} \sum_{n_2} \Big[ x_{3m_1 n_1} x_{1m_2 n_2} C_{mn}^{m_1 n_1 m_2 n_2} + $$
$$+ x_{3m_1 n_1} x_{2m_2 n_2} D_{mn}^{m_1 n_1 m_2 n_2} \Big],$$

where

$$A_{mn}^{m_1 n_1 m_2 n_2} = m_1 \pi^3 \left( m_2^2 + \frac{1-\mu}{2} \beta_1^2 n_2^2 \right) a_{1mn}^{m_1 n_1 m_2 n_2} -$$
$$- \frac{1+\mu}{2} \pi^3 \beta_1^2 n_1 m_2 n_2 a_{2mn}^{m_1 n_1 m_2 n_2},$$

$$B_{mn}^{m_1 n_1 m_2 n_2} = \beta_1 n_1 \pi^3 \left( \beta_1^2 n_2^2 + \frac{1-\mu}{2} m_2^2 \right) a_{3mn}^{m_1 n_1 m_2 n_2} -$$
$$- \frac{1+\mu}{2} \pi^3 \beta_1 m_1 m_2 n_2 a_{4mn}^{m_1 n_1 m_2 n_2},$$

$$C_{mn}^{m_1 n_1 m_2 n_2} = \pi^3 m_2 \left( m_1^2 + \mu \beta_1^2 n_1^2 \right) a_{5mn}^{m_1 n_1 m_2 n_2} +$$
$$+ (1-\mu) \pi^3 \beta_1^2 m_1 n_1 n_2 a_{6mn}^{m_1 n_1 m_2 n_2}$$
$$- \pi^3 m_1 \left( m_2^2 + \frac{1-\mu}{2} \beta_1^2 n_2^2 \right) a_{7mn}^{m_1 n_1 m_2 n_2} -$$
$$- \frac{1+\mu}{2} \pi^3 \beta_1^2 n_1 m_2 n_2 a_{8mn}^{m_1 n_1 m_2 n_2},$$

$$D_{mn}^{m_1 n_1 m_2 n_2} = \pi^3 \beta_1 n_2 \left( \beta_1^2 n_1^2 + \mu m_1^2 \right) a_{5mn}^{m_1 n_1 m_2 n_2} +$$
$$+ (1-\mu) \pi^3 \beta_1 m_1 n_1 m_2 a_{6mn}^{m_1 n_1 m_2 n_2} -$$
$$- \pi^3 \beta_1 n_1 \left( \beta_1^2 n_2^2 + \frac{1-\mu}{2} m_2^2 \right) a_{8mn}^{m_1 n_1 m_2 n_2} -$$
$$- \frac{1+\mu}{2} \pi^3 \beta_1 m_1 m_2 n_2 a_{7mn}^{m_1 n_1 m_2 n_2},$$

$$a_{1mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \sin \pi m_1 x \sin \pi n_1 x \cos \pi m_2 x \times$$
$$\times \sin \pi n_2 y \cos \pi m x \sin \pi n y \, dx dy,$$

$$a_{2mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \cos \pi m_1 x \cos \pi n_1 x \sin \pi m_2 x \times$$
$$\times \cos \pi n_2 y \cos \pi m x \sin \pi n y \, dx dy,$$

$$a_{3mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \sin \pi m_1 x \cos \pi n_1 x \sin \pi m_2 x \times$$
$$\times \sin \pi n_2 y \sin \pi m x \cos \pi n y \, dx dy,$$

$$a_{4mn}^{m_1 n_1 m_2 n_2} = \int_0^1 \int_0^1 \cos \pi m_1 x \sin \pi n_1 x \cos \pi m_2 x \times$$
$$\times \cos \pi n_2 y \sin \pi m x \cos \pi n y \, dx dy,$$

$$a_{5mn}^{m_1n_1m_2n_2} = \int_0^1\int_0^1 \sin\pi m_1x\sin\pi n_1x\sin\pi m_2x\times$$
$$\times\sin\pi n_2y\sin\pi mx\sin\pi ny\mathrm{d}x\mathrm{d}y,$$

$$a_{6mn}^{m_1n_1m_2n_2} = \int_0^1\int_0^1 \cos\pi m_1x\cos\pi n_1x\cos\pi m_2x\times$$
$$\times\cos\pi n_2y\sin\pi mx\sin\pi ny\mathrm{d}x\mathrm{d}y,$$

$$a_{7mn}^{m_1n_1m_2n_2} = \int_0^1\int_0^1 \cos\pi m_1x\sin\pi n_1x\cos\pi m_2x\times$$
$$\times\sin\pi n_2y\sin\pi mx\sin\pi ny\mathrm{d}x\mathrm{d}y,$$

$$a_{8mn}^{m_1n_1m_2n_2} = \int_0^1\int_0^1 \sin\pi m_1x\cos\pi n_1x\sin\pi m_2x\times$$
$$\times\cos\pi n_2y\sin\pi mx\sin\pi ny\mathrm{d}x\mathrm{d}y.$$

The analysis of the structure of equations (18)-(22) shows that equations (18) and (19) are coupled with each other via linear terms and with equation (20) in terms of nonlinear terms $F_{jmn}(j = 1,2,3)$ . Equations (21) and (22) are coupled with each other and with Eq. (20) only via linear terms. Thus, the linearized equations (18)-(22) are decoupled in two linear subsystems.

### 3.1. Solution of the eigen value problem and decoupling the equations of motion

To determine the natural frequencies of linear vibrations $\omega_{imn}$ ($i$ = 1,2,3,4,5), it is a need to solve the linear eigen value problem. The characteristic equation of the linearized equations (18) and (19) has the form

$$\omega_{mn}^4 - \omega_{mn}^2(S_{11}^{mn} + S_{22}^{mn}) + S_{11}^{mn}S_{22}^{mn} - S_{12}^{mn}S_{21}^{mn} = 0, \quad (25)$$

the solution of which gives the natural frequencies of in-plane vibrations

$$\omega_{1mn}^2 = \pi^2\left(m^2 + \beta_1^2 n^2\right),$$
$$\omega_{2mn}^2 = \frac{1-\mu}{2}\pi^2\left(m^2 + \beta_1^2 n^2\right), \quad (26)$$

which coincide with those obtained in [16,19].
The linearized set of equations (20)-(22) provides the following frequency equation:

$$\omega_{mn}^6 + e_2^{mn}\omega_{mn}^4 + e_1^{mn}\omega_{mn}^2 + e^{mn} = 0, \quad (27)$$

where

$$e^{mn} = s_{33}^{mn}s_{45}^{mn}s_{54}^{mn} + s_{44}^{mn}s_{35}^{mn}s_{53}^{mn} + s_{55}^{mn}s_{34}^{mn}s_{43}^{mn} -$$
$$-s_{33}^{mn}s_{44}^{mn}s_{55}^{mn} - s_{34}^{mn}s_{53}^{mn}s_{54}^{mn} - s_{43}^{mn}s_{35}^{mn}s_{45}^{mn},$$
$$e_1^{mn} = s_{33}^{mn}s_{44}^{mn} + s_{33}^{mn}s_{55}^{mn} + s_{44}^{mn}s_{55}^{mn} - s_{34}^{mn}s_{43}^{mn} -$$
$$-s_{35}^{mn}s_{53}^{mn} - s_{45}^{mn}s_{54}^{mn},$$
$$e_2^{mn} = -s_{33}^{mn} - s_{44}^{mn} - s_{55}^{mn}.$$

The solution of equation (27) results in three sets of natural frequencies, $\omega_{3mn}$, $\omega_{4mn}$ and $\omega_{5mn}$, and the least of them, $\omega_{3mn}$, corresponds to the frequency of flexural vibrations. It is defined as

$$\omega_{3mn}^2 = \frac{1}{4\beta_2^2}\{12k^2(1-\mu)+$$
$$+\beta_2^2\pi^2\left(2+k^2(1-\mu)\right)\left(m^2+\beta_1^2n^2\right)-$$
$$-\left[\left[12k^2(1-\mu)+\beta_2^2\pi^2\left(2+k^2(1-\mu)\right)\left(m^2+\beta_1^2n^2\right)\right]^2-\right.$$
$$\left.-8\beta_2^4k^2(1-\mu)\pi^4\left(m^2+\beta_1^2n^2\right)^2\right]^{1/2}\Bigg\}. \quad (28)$$

The other two roots of equation (27) correspond to the high frequency vibrations and have the form

$$\omega_{4mn}^2 = \frac{1-\mu}{2}\left[\frac{12}{\beta_2^2}k^2 + \pi^2\left(m^2+\beta_1^2n^2\right)\right], \quad (29)$$

$$\omega_{5mn}^2 = \frac{1}{4\beta_2^2}\{12k^2(1-\mu)+$$
$$+\beta_2^2\pi^2\left(2+k^2(1-\mu)\right)\left(m^2+\beta_1^2n^2\right)+$$
$$+\left[\left[12k^2(1-\mu)+\beta_2^2\pi^2\left(2+k^2(1-\mu)\right)\left(m^2+\beta_1^2n^2\right)\right]^2\right.$$
$$\left.-8\beta_2^4k^2(1-\mu)\pi^4\left(m^2+\beta_1^2n^2\right)^2\right]^{1/2}\Bigg\}. \quad (30)$$

The natural frequencies correspond to mutually orthogonal eigen vectors

$$L_{mn}^{\mathrm{I}}\left\{L_{imn}^{\mathrm{I}}\right\},\quad L_{mn}^{\mathrm{II}}\left\{L_{imn}^{\mathrm{II}}\right\}\quad (i=1,2), \quad (31)$$

$$L_{mn}^{\mathrm{III}}\left\{L_{imn}^{\mathrm{III}}\right\},\quad L_{mn}^{\mathrm{IV}}\left\{L_{imn}^{\mathrm{IV}}\right\},\quad L_{mn}^{\mathrm{V}}\left\{L_{imn}^{\mathrm{V}}\right\}\quad (i=3,4,5). \quad (32)$$

Following [20], let us expand the matrices $S_{ij}^{mn}$ ($i$, $j$ = 1,2), $S_{ij}^{mn}$ ($i$,$j$ = 3,4,5) and generalized displacements $x_{imn}$ entering in equations (18)-(22) in terms of the eigen vectors (31) and (32)

$$S_{ij}^{mn} = \omega_{1mn}^2 L_{imn}^{I} L_{jmn}^{I} + \omega_{2mn}^2 L_{imn}^{II} L_{jmn}^{II},$$
$$x_{imn} = X_{1mn} L_{imn}^{I} + X_{2mn} L_{imn}^{II} \qquad (i = 1,2), \tag{33}$$

$$s_{ij}^{mn} = \omega_{3mn}^2 L_{imn}^{III} L_{jmn}^{III} + \omega_{4mn}^2 L_{imn}^{IV} L_{jmn}^{IV} + \omega_{5mn}^2 L_{imn}^{V} L_{jmn}^{V}, \tag{34}$$

$$x_{imn} = X_{3mn} L_{imn}^{III} + X_{4mn} L_{imn}^{IV} + X_{5mn} L_{imn}^{V} \qquad (i = 3,4,5) \tag{35}$$

Now substituting expansions (33)-(35) in Eqs. (18)-(22) and then multiplying (18)-(19) successively by $L_{imn}^{I}$, $L_{imn}^{II}$, and (20)-(22) successively by $L_{imn}^{III}$, $L_{imn}^{IV}$, and finally by $L_{imn}^{V}$ with due account for the conditions of orthogonality of the eigen vectors

$$\begin{aligned} L_{imn}^{K} L_{imn}^{N} &= 0 \quad \text{at} \quad K \neq N \\ L_{imn}^{K} L_{imn}^{K} &= 1 \quad (K, N{=}I,II,III,IV,V), \end{aligned} \tag{36}$$

we are led to the following set of equations of motion:

$$\ddot{X}_{1mn} + \chi_1 D^{\gamma} X_{1mn} + \omega_{1mn}^2 X_{1mn} = -\sum_i^2 F_{imn} L_{imn}^{I}, \tag{37}$$

$$\ddot{X}_{2mn} + \chi_2 D^{\gamma} X_{2mn} + \omega_{2mn}^2 X_{2mn} = -\sum_i^2 F_{imn} L_{imn}^{II}, \tag{38}$$

$$\ddot{X}_{3mn} + \chi_3 D^{\gamma} X_{3mn} + \omega_{3mn}^2 X_{3mn} = -F_{3mn} L_{3mn}^{III}, \tag{39}$$

$$\ddot{X}_{4mn} + \chi_4 D^{\gamma} X_{4mn} + \omega_{4mn}^2 X_{4mn} = 0, \tag{40}$$

$$\ddot{X}_{5mn} + \chi_5 D^{\gamma} X_{5mn} + \omega_{5mn}^2 X_{5mn} = 0, \tag{41}$$

in terms of new generalized displacements $X_{jmn}$

$$X_{1mn} = x_{1mn} L_{1mn}^{I} + x_{2mn} L_{2mn}^{I}, \tag{42}$$

$$X_{2mn} = x_{1mn} L_{1mn}^{II} + x_{2mn} L_{2mn}^{II}, \qquad (42)$$

$$X_{3mn} = x_{3mn} L_{3mn}^{III} + x_{4mn} L_{2mn}^{III} + x_{5mn} L_{5mn}^{III}, \tag{43}$$

$$X_{4mn} = x_{3mn} L_{3mn}^{IV} + x_{4mn} L_{2mn}^{IV} + x_{5mn} L_{5mn}^{IV}, \tag{44}$$

$$X_{5mn} = x_{3mn} L_{3mn}^{V} + x_{4mn} L_{2mn}^{V} + x_{5mn} L_{5mn}^{V}. \tag{45}$$

It should be emphasized that the left-hand side parts of (37)-(41) are linear and independent of each other, while equations (37)-(39) are coupled only by non-linear terms in their right-hand sides.

Moreover, the set of equations (37)-(41) is decoupled into three subsystems, namely: the first subset compiles three nonlinear fractional derivative equations (37)-(39), the second and the third subsystems involve one linear fractional derivative equation each, i.e. equations (40) and (41), respectively. Thus, in order to find a solution, it is need to examine each subsystem.

### 3.2. Analysis of the reduced equations of motion

Equations (40) and (41) describe free damped vibrations of a linear oscillator with a viscoelastic resistance force modelled in terms of the fractional derivative Kelvin-Voigt model [24]. For the case of weak damping, i.e. when $\chi_i = \varepsilon æ_i$ or $\chi_i = \varepsilon^2 æ$ with $0 < \varepsilon = 1$, approximate analytical solutions of equations similar to (40) and (41) have been found in [28,52] utilyzing the fractional derivative expansion method [27], which is the extension of the multiple time scales procedure [53]. The case of $\varepsilon$ -order damping and the half-derivative, i.e. when the order of the fractional derivative is $\gamma = 1/2$, was treated in [54] using the averaging perturbation technique.

Free damped vibrations of a linear fractional derivative Kelvin-Voigt oscillator in a medium with finite viscosity, i.e. without any restrictions on the magnitude of the damping coefficient $\chi_i$, have been studied analytically in [24,52] utilizing the construction of the Green function, which was proposed for the first time for such fractional derivative equations by Professor Yury Rossikhin in his PhD thesis [55] in 1970 and then published in 1971 in the pioneer paper [56]. Further this procedure was generalized for dynamics of linear oscillators, beams, plates and shells using different fractional operator models, and their overview could be found in [24,28,29].

As for the first subsystem (37)-(39) involving three nonlinear equations with fractional derivative terms, then it has the similar structure as the set of three governing equations considered previously but ignoring the influence of the rotary inertia and shear deformations [19].

Following [19,20] it could be shown that the solution of equations (37)-(39) could be constructed using the generalized method of multiple time scales suggested in [27]. We will not repeat this procedure, since it is described in detail in [20,57], and it could be easily adopted to equations (37)-(39) within an accuracy of coefficients.

Thus, it has been revealed that nonlinear vibrations of the plate could be accompanied by different types of the internal resonance when two or more modes could be coupled, resulting in the energy exchange between the coupled modes. Moreover, its type depends on the order of smallness of the viscosity involved into consideration. Thus, it has been found that at the $\varepsilon$ – order, damped vibrations could be accompanied by the following types of the internal resonance:

the two-to-one internal resonance (2:1), when one natural frequency is twice the other natural frequency,

$$\omega_1 = 2\omega_3 \; (\omega_1 \neq \omega_2,\, 2\omega_3 \neq \omega_2), \tag{47}$$

$$\omega_2 = 2\omega_3 \; (\omega_1 \neq \omega_2,\, 2\omega_3 \neq \omega_1), \tag{48}$$

the one-to-one-to-two internal resonance (1:1:2), that is,

$$\omega_1 = \omega_2 = 2\omega_3\,; \tag{49}$$

at the $\varepsilon^2$ -order, damped vibrations could be accompanied by the following types of the internal resonance:

the one-to-one internal resonance (1:1)

$$\omega_1 = \omega_2 \; (\omega_3 \neq \omega_1,\, \omega_3 \neq \omega_2), \tag{50}$$

$$\begin{aligned} &\omega_1 = \omega_3 \; (\omega_2 \neq \omega_1,\, \omega_2 \neq \omega_3), \\ &\omega_2 = \omega_3 \; (\omega_1 \neq \omega_2,\, \omega_1 \neq \omega_3), \end{aligned} \tag{51}$$

the one-to-one-to-one internal resonance (1:1:1)

$$\omega_1 = \omega_2 = \omega_3\,, \tag{52}$$

the combinational resonance of the additive-difference type

$$\omega_1 = \omega_2 + 2\omega_3\,, \tag{53}$$

$$\begin{aligned} &\omega_1 = 2\omega_3 - \omega_2\,, \\ &\omega_1 = \omega_2 - 2\omega_3\,, \end{aligned} \tag{54}$$

where $\omega_1$ and $\omega_2$ are the frequencies of certain modes of in-plane vibrations in the $x$- and $y$- axes, respectively, and $\omega_3$ is the frequency of a certain mode of out-of-plane vibrations.

For each type of the resonance, the nonlinear sets of resolving equations in terms of amplitudes and phase differences could be obtained using the same procedure as in [20]. The influence of viscosity on the energy exchange mechanism is revealed by the fact that each mode is characterized by its damping coefficient connected with the natural frequency by the exponential relationship with a negative fractional exponent. Thus, during free vibrations of the plate with internal resonances three regimes could be observed: stationary (absence of damping at $\gamma = 0$), quasistationary (damping is defined by an ordinary derivative at $\gamma = 1$), and transient (damping is defined by a fractional derivative at $0 < \gamma < 1$).

## 4. ANALYSIS OF SPECTRA OF NATURAL FREQUENCIES

In order to show that the phenomenon of internal resonance could be very critical, since in the thin plate under consideration the internal resonance is always present, it is a need to analyze the spectra of natural frequencies.

Thus, natural frequencies of vibrations $\omega_{imn}$ (i = 1,2,...,5) calculated according to (26) and (28)-(30), as well as frequency of vertical flexural vibrations without shear deformations and rotary inertia calculated via the formula [20]

$$\bar{\omega}_{3mn}^2 = \frac{\beta_2^2}{12}\pi^4\left(m^2 + \beta_1^2 n^2\right)^2 \tag{55}$$

are given in Tables 1-3 for a square plate, i.e. at $\beta_1 = a/b = 1$, at $\beta_2 = h/a = 0.1$ and 0.025, respectively.

Reference to Tables 1-3 shows the influence of the shear deformations and rotary inertia on the frequencies of flexural vibrations, in so doing the thicker the plate, the more difference between the frequencies $\omega_3$ and $\bar{\omega}_3$. Thus, for example, for the square plate the frequency of the fundamental mode at $m = 1$, $n = 1$ calculated by the classical theory at $\beta_2 = 0.1$, 0.05 and 0.025 is reduced, respectively, by 3.51, 1.05 and 0.7% as compared with that calculated by the refined theory. This difference increases for more high frequencies, what is evident from Table 4. Natural frequencies for a rectangular plate at $\beta_1 = 0.5$ and $\beta_2 = 0.05$ are presented in Table 5. The influence of the ratio of the plate's dimensions on natural frequencies is seen from Table 6, whence it follows that the difference between the frequencies according to classical and refined theories increases with the increase in plate's length.

From Tables 1-3 and 5 it is seen that the internal resonances of all types (47)-(54) could take place, and the occurrence of this or that case depends on the dimensions of the plate, i.e. on magnitudes of the coefficients $\beta_1$ and $\beta_2$.

As soon as the case of the internal resonance is revealed, then the further treatment of nonlinear

Table 1. Natural frequencies of vibrations $\omega_{imn}$ $(i=1,2,...,5)$ at $\beta_1=1$ and $\beta_2=0.1$.

| $m$ | $n$ | $\omega_{1mn}$ | $\omega_{2mn}$ | $\omega_{3mn}/\bar{\omega}_{3mn}$ | $\omega_{4mn}$ | $\omega_{5mn}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 4.443 | 2.628 | 0.550/0.570 | 18.892 | 19.370 |
| 1 | 2 | 7.023 | 4.156 | 1.313/1.425 | 19.164 | 20.298 |
| 2 | 1 | 7.023 | 4.156 | 1.313/1.425 | 19.164 | 20.298 |
| 2 | 2 | 8.886 | 5.257 | 2.017/2.279 | 19.433 | 21.164 |
| 1 | 3 | 9.935 | 5.877 | 2.458/2.849 | 19.610 | 21.715 |
| 3 | 1 | 9.935 | 5.877 | 2.458/2.849 | 19.610 | 21.715 |
| 2 | 3 | 11.327 | 6.701 | 3.080/3.704 | 19.873 | 22.500 |
| 3 | 3 | 13.329 | 7.885 | 4.043/5.128 | 20.302 | 23.731 |
| 1 | 4 | 12.953 | 7.663 | 3.857/4.843 | 20.217 | 23.491 |
| 2 | 4 | 14.050 | 8.312 | 4.405/5.698 | 20.472 | 24.198 |
| 3 | 4 | 15.708 | 9.293 | 5.263/7.123 | 20.889 | 25.318 |
| 4 | 4 | 17.772 | 10.514 | 6.368/9.117 | 21.460 | 26.784 |
| 1 | 5 | 16.019 | 9.471 | 5.427/7.408 | 20.972 | 25.534 |
| 2 | 5 | 16.918 | 10.009 | 5.907/8.262 | 21.217 | 26.169 |
| 3 | 5 | 18.319 | 10.831 | 6.667/9.687 | 21.621 | 27.184 |
| 4 | 5 | 20.116 | 11.901 | 7.660/11.681 | 22.173 | 28.531 |
| 5 | 5 | 22.214 | 13.142 | 8.838/14.246 | 23.510 | 30.155 |

Table 2. Natural frequencies of vibrations $\omega_{imn}$ $(i=1,2,...,5)$ at $\beta_1=1$ and $\beta_2=0.05$.

| $m$ | $n$ | $\omega_{1mn}$ | $\omega_{2mn}$ | $\omega_{3mn}/\bar{\omega}_{3mn}$ | $\omega_{4mn}$ | $\omega_{5mn}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 4.443 | 2.628 | 0.282/0.285 | 37.509 | 37.755 |
| 1 | 2 | 7.023 | 4.156 | 0.697/0.712 | 37.647 | 38.253 |
| 2 | 2 | 8.886 | 5.257 | 1.101/1.140 | 37.784 | 38.740 |
| 1 | 3 | 9.935 | 5.877 | 1.365/1.423 | 37.915 | 39.060 |
| 2 | 3 | 11.327 | 6.701 | 1.753/1.852 | 38.012 | 39.530 |
| 3 | 3 | 13.329 | 7.885 | 2.381/2.564 | 38.238 | 40.296 |
| 1 | 4 | 12.953 | 7.663 | 2.257/2.422 | 38.193 | 40.145 |
| 2 | 4 | 14.050 | 8.312 | 2.627/2.849 | 38.329 | 40.596 |
| 3 | 4 | 15.708 | 9.293 | 3.224/3.561 | 38.553 | 41.332 |
| 4 | 4 | 17.772 | 10.514 | 4.030/4.559 | 38.866 | 42.329 |
| 1 | 5 | 16.019 | 9.471 | 3.341/3.704 | 38.598 | 41.476 |
| 2 | 5 | 16.918 | 10.009 | 3.689/4.131 | 38.732 | 41.906 |
| 3 | 5 | 18.319 | 10.831 | 4.253/4.843 | 38.954 | 42.607 |
| 4 | 5 | 20.116 | 11.901 | 5.017/5.841 | 39.264 | 43.560 |
| 5 | 5 | 22.214 | 13.142 | 5.956/7.123 | 39.658 | 44.743 |

Table 3. Natural frequencies of vibrations $\omega_{imn}$ $(i = 1, 2, ..., 5)$ at $\beta_1 = 1$ and $\beta_2 = 0.025$.

| $m$ | $n$ | $\omega_{1mn}$ | $\omega_{2mn}$ | $\omega_{3mn} / \bar{\omega}_{3mn}$ | $\omega_{4mn}$ | $\omega_{5mn}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 4.443 | 2.628 | 0.142/0.143 | 74.879 | 75.006 |
| 1 | 2 | 7.023 | 4.156 | 0.354/0.356 | 74.948 | 75.257 |
| 2 | 2 | 8.886 | 5.257 | 0.565/0.570 | 75.018 | 75.509 |
| 1 | 3 | 9.935 | 5.877 | 0.685/0.712 | 75.064 | 75.677 |
| 2 | 3 | 11.327 | 6.701 | 0.913/0.926 | 75.133 | 75.927 |
| 3 | 3 | 13.329 | 7.885 | 1.257/1.282 | 75.247 | 76.341 |
| 1 | 4 | 12.953 | 7.663 | 1.188/1.210 | 75.225 | 76.258 |
| 2 | 4 | 14.050 | 8.312 | 1.394/1.425 | 75.293 | 76.505 |
| 3 | 4 | 15.708 | 9.293 | 1.732/1.781 | 75.408 | 76.914 |
| 4 | 4 | 17.772 | 10.514 | 2.201/2.279 | 75.568 | 77.480 |
| 1 | 5 | 16.019 | 9.471 | 1.800/1.852 | 75.431 | 76.995 |
| 2 | 5 | 16.918 | 10.009 | 2.001/2.066 | 75.500 | 77.238 |
| 3 | 5 | 18.319 | 10.831 | 2.334/2.422 | 75.614 | 77.640 |
| 4 | 5 | 20.116 | 11.901 | 2.795/2.920 | 75.774 | 78.198 |
| 5 | 5 | 22.214 | 13.142 | 3.378/3.561 | 75.979 | 78.905 |

Table 4. Difference in vertical frequencies of flexural vibrations $\delta = \left[ (\omega_3 - \bar{\omega}_3) / \bar{\omega}_3 \right] 100\%$ at $\beta_1 = 1$ for plates of different thickness.

| | $m=1,\ n=1$ | | | $m=5,\ n=5$ | | |
|---|---|---|---|---|---|---|
| $\beta_2$ | 0.1 | 0.05 | 0.025 | 0.1 | 0.05 | 0.025 |
| $\delta$, % | 3.51 | 1.05 | 0.70 | 61.19 | 16.38 | 5.14 |

equations (37)-(39) could be carried out by the procedure developed in [27] within an accuracy of the coefficients.

## CONCLUSION

In the present paper, the nonlinear free vibrations of fractionally damped plates are studied, equations of motion of which take the rotary inertia and shear deformations into account and involve five coupled nonlinear differential equations in terms of three mutually orthogonal displacements and two angles of rotation. The procedure resulting in decoupling linear parts of equations has been adopted with further utilization of the generalized method of multiple time scales for solving nonlinear governing equations of motion, in so doing the amplitude functions have been expanded into power series in terms of the small parameter and depend on different time scales. Numerical analysis of the natural frequency spectra reveals the possibility of the occurrence of different internal and combinational resonances.


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Table 5. Natural frequencies of vibrations $\omega_{imn}$ $(i=1,2,...,5)$ at $\beta_1=0.5$ and $\beta_2=0.05$.

| $m$ | $n$ | $\omega_{1mn}$ | $\omega_{2mn}$ | $\omega_{3mn}/\bar{\omega}_{3mn}$ | $\omega_{4mn}$ | $\omega_{5mn}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 3.685 | 2.531 | 0.177/0.178 | 37.474 | 37.629 |
| 1 | 2 | 5.211 | 3.580 | 0.282/0.285 | 37.509 | 37.755 |
| 2 | 1 | 6.550 | 4.129 | 0.594/0.605 | 37.612 | 38.129 |
| 2 | 2 | 7.370 | 5.062 | 0.697/0.712 | 37.647 | 38.253 |
| 1 | 3 | 7.171 | 4.675 | 1.266/1.318 | 37.841 | 38.940 |
| 3 | 1 | 9.602 | 5.860 | 0.456/0.463 | 37.566 | 37.963 |
| 2 | 3 | 8.714 | 6.109 | 0.866/0.830 | 37.704 | 38.457 |
| 3 | 2 | 10.142 | 6.611 | 1.365/1.425 | 37.875 | 39.059 |
| 3 | 3 | 11.055 | 7.593 | 1.528/1.603 | 37.932 | 39.257 |
| 1 | 4 | 9.264 | 5.839 | 0.697/0.712 | 37.647 | 38.253 |
| 4 | 1 | 12.699 | 7.653 | 2.164/2.315 | 38.159 | 40.031 |
| 2 | 4 | 10.423 | 7.159 | 1.101/1.140 | 37.784 | 38.740 |
| 4 | 2 | 13.101 | 8.258 | 2.257/2.422 | 38.193 | 40.145 |
| 3 | 4 | 12.324 | 8.639 | 1.753/1.852 | 38.012 | 39.530 |
| 4 | 3 | 13.780 | 9.125 | 2.417/2.600 | 38.250 | 40.334 |
| 4 | 4 | 14.740 | 10.125 | 2.626/2.849 | 38.329 | 40.596 |
| 1 | 5 | 11.409 | 7.051 | 1.001/1.033 | 37.750 | 38.619 |
| 5 | 1 | 15.814 | 9.470 | 3.253/3.597 | 38.564 | 41.368 |
| 2 | 5 | 12.330 | 8.236 | 1.397/1.460 | 37.887 | 39.099 |
| 5 | 2 | 16.134 | 9.971 | 3.341/3.704 | 38.598 | 41.476 |
| 3 | 5 | 13.880 | 9.686 | 2.038/2.172 | 38.114 | 39.878 |
| 5 | 3 | 16.674 | 10.726 | 3.487/3.882 | 38.654 | 41.656 |
| 4 | 5 | 15.969 | 11.169 | 2.898/3.170 | 38.430 | 40.930 |
| 5 | 4 | 17.437 | 11.647 | 3.688/4.131 | 38.732 | 41.906 |
| 5 | 5 | 18.425 | 12.656 | 3.945/4.452 | 38.832 | 42.223 |

Table 6. Difference in vertical frequencies of flexural vibrations $\delta=\left[\left(\omega_3-\bar{\omega}_3\right)/\bar{\omega}_3\right]100\%$ at $\beta_2=0.05$ for plates of different length.

| | $m=1,\ n=1$ | | | $m=5,\ n=5$ | | |
|---|---|---|---|---|---|---|
| $\beta_1$ | 0.5 | 1 | 2 | 0.5 | 1 | 2 |
| $\delta$, % | 0.56 | 1.05 | 2.11 | 11.39 | 16.38 | 29.75 |


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# A.A. ILYUSHIN'S FINAL RELATION, ALTERNATIVE EQUIVALENT RELATIONS AND VERSIONS OF ITS APPROXIMATION IN PROBLEMS OF ELASTIC DEFORMATION OF PLATES AND SHELLS
# PART 2: ALTERNATIVE EQUIVALENT RELATIONS OF A.A. ILYUSHIN

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**Abstract:** The finite relationship between the forces and moments of plates and shells in the parametric form of the theory of small elastoplastic deformations is investigated of A.A. Ilyushin, to determine the load-bearing capacity of structures from a material without hardening. A geometric image of the exact yield surface in the space of generalized stresses is obtained. In the first part of the article the conclusion of the final relation is given. In the second and third parts, by introducing other parameters, alternative equivalent dependences of the final relationship have been developed and variants of its approximation for application in computational practice are considered. In the fourth part, additional properties of the final relationship are considered, the possibility and necessity of its use in problems of plastic deformation of plates and shells is shown.

**Keywords**: the plasticity theory, plastic deformation of plates and shells, a surface of fluidity, a plasticity condition.

# КОНЕЧНОЕ СООТНОШЕНИЕ А.А. ИЛЬЮШИНА, АЛЬТЕРНАТИВНЫЕ ЭКВИВАЛЕНТНЫЕ ЗАВИСИМОСТИ И ВАРИАНТЫ ЕГО АППРОКСИМАЦИИ В ЗАДАЧАХ ПЛАСТИЧЕСКОГО ДЕФОРМИРОВАНИЯ ПЛАСТИН И ОБОЛОЧЕК
# ЧАСТЬ 2: АЛЬТЕРНАТИВНЫЕ ЭКВИВАЛЕНТНЫЕ ЗАВИСИМОСТИ КОНЕЧНОГО СООТНОШЕНИЯ А.А. ИЛЬЮШИНА

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**Аннотация**: Выполнено исследование конечного соотношения между силами и моментами пластин и оболочек в параметрическом виде теории малых упругопластических деформаций А.А. Ильюшина, для определения несущей способности конструкций из материала без упрочнения. Получен геометрический образ точной поверхности текучести в пространстве обобщенных напряжений. В первой части статьи приводится вывод конечного соотношения. Во второй и третьей частях введением других параметров разработаны альтернативные эквивалентные зависимости конечного соотношения и рассмотрены варианты его аппроксимации для применения в расчетной практике. В четвертой части рассмотрены дополнительные свойства конечного соотношения, показана возможность и необходимость его использования в задачах пластического деформирования пластин и оболочек.

**Ключевые слова**: теория пластичности, пластическое деформирование пластин и оболочек, поверхность текучести, условия пластичности.

### 2.1. Alternative equivalent relations of a final relation

In the work [9], in integrating the integrals (4.25), integration over the intensity of the deformations $e_i$ is performed instead of integrating over the coordinate $z$. Let us show that we can obtain an alternative finite relation by calculating the integrals (4.25) with respect to the coordinate $z$, and compare the results of the calculations.

Intensity of deformations, according to (4.7) [9]:

$$e_i = \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon - 2zP_{\varepsilon\chi} + z^2 P_\chi},$$
$$P_\varepsilon = \varepsilon_1^2 + \varepsilon_1\varepsilon_2 + \varepsilon_2^2 + \varepsilon_{12}^2,\ P_\chi = \chi_1^2 + \chi_1\chi_2 + \chi_2^2 + \chi_{12}^2,$$
$$P_{\varepsilon\chi} = \varepsilon_1\chi_1 + \varepsilon_2\chi_2 + \frac{1}{2}\varepsilon_1\chi_2 + \frac{1}{2}\varepsilon_2\chi_1 + \varepsilon_{11}\chi_{12}. \tag{2.1}$$

Let's consider values of intensity of deformations in three points disposed on an axis $z$

$z = -\frac{h}{2},\ z = +\frac{h}{2},\ z = 0.$ Let's designate them accordingly:

$$e_{i1} = \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi}\ \left(z = -\frac{h}{2}\right),$$
$$e_{i2} = \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi}\ \left(z = +\frac{h}{2}\right), \tag{2.2}$$
$$e_{i0} = \frac{2}{\sqrt{3}}\sqrt{P_\varepsilon}\ (z = 0).$$

Considering the last as the equations concerning three quadratic forms $P_\chi$, $P_{\varepsilon\chi}$, $P_\varepsilon$, we copy them in a kind:

$$P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi = \frac{3}{4}e_{i1}^2,$$
$$P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi = \frac{3}{4}e_{i2}^2,\ P_\varepsilon = \frac{3}{4}e_{i0}^2. \tag{2.3}$$

Solving them with respect to quadratic forms leads to the following results:

$$P_\varepsilon = \frac{3}{4}e_{i0}^2,\ hP_{\varepsilon\chi} = \frac{3}{8}\left(e_{i1}^2 - e_{i2}^2\right),$$
$$\frac{h^2}{4}P_\chi = \frac{3}{16}\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right). \tag{2.4}$$

We introduce two basic parameters λ and μ:

$$\lambda = \frac{e_{i2}}{e_{i1}},\ \mu = \frac{e_{i0}}{e_{i1}}. \tag{2.5}$$

These parametres satisfy to conditions: $0 \le \lambda \le 1$, $0 \le \mu \le 1$ as $e_{i1}$ – there is a maximum value of intensity of deformations, if $P_{\varepsilon\chi} < 0$. Then formulas (2.3) can be copied in a kind:

$$P_\varepsilon = \frac{3}{4}\mu^2 e_{i1}^2,\ hP_{\varepsilon\chi} = \frac{3}{8}\left(1 - \lambda^2\right)e_{i1}^2,$$
$$\frac{h^2}{4}P_\chi = \frac{3}{16}\left(2 + 2\lambda^2 - 4\mu^2\right)e_{i1}^2. \tag{2.6}$$

In formulas (4.23')-(4.24') [9], there are three types of integrals that are common in shell thickness:

$$J_1 = \frac{\sqrt{3}}{2}\sigma_s\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{dz}{X^{\frac{1}{2}}},\ J_2 = \frac{\sqrt{3}}{2}\sigma_s\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{zdz}{X^{\frac{1}{2}}},$$
$$J_3 = \frac{\sqrt{3}}{2}\sigma_s\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{z^2dz}{X^{\frac{1}{2}}},\ X^{\frac{1}{2}} = \sqrt{c + bz + az^2}, \tag{2.7}$$
$$c = P_\varepsilon,\ b = -2P_{\varepsilon\chi},\ a = P_\chi.$$

These integrals tabular. According to formulas (380.001, 380.011, 380.021) [42]

$$J_1 = \frac{\sqrt{3}}{2}\sigma_s\left[\frac{1}{\sqrt{a}}\ln\left|2\sqrt{a}X^{\frac{1}{2}} + 2az + b\right|\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$
$$J_2 = \frac{\sqrt{3}}{2}\sigma_s\left[\frac{X^{\frac{1}{2}}}{a} - \frac{b}{2a}\int\frac{dz}{X^{\frac{1}{2}}}\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$
$$J_3 = \frac{\sqrt{3}}{2}\sigma_s\left[\left(\frac{z}{2a} - \frac{3b}{4a^2}\right)X^{\frac{1}{2}} + \left(\frac{3b^2 - 4ac}{8a^2}\right)\int\frac{dz}{X^{\frac{1}{2}}}\right]_{-\frac{h}{2}}^{\frac{h}{2}}, \tag{2.8}$$
$$X^{\frac{1}{2}} = \sqrt{c + bz + az^2},\ c = P_\varepsilon,\ b = -2P_{\varepsilon\chi},\ a = P_\chi.$$

As well as in [9], we will consider that tensile deformation and shift of a middle surface $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_{12}$ are commensurable or small compared with bending strains of a shell $\pm\frac{h}{2}\chi_1$, $\pm\frac{h}{2}\chi_2$, $\pm\frac{h}{2}\chi_{12}$ or that the last are dominating if the point $z_0$ (minimum) $e_i$ does not fall outside the limits a thickness of a shell, i.e. if $-\frac{h}{2} \le z_0 = \frac{P_{\varepsilon\chi}}{P_\chi} \le \frac{h}{2}$.

Deformations of a middle surface we will name large or dominating compared with bending strains if the point $z_0$ is disposed out of a thickness of a shell i.e. if one of inequalities takes place $z_0 = \frac{P_{\varepsilon\chi}}{P_\chi} > \frac{h}{2}$, $z_0 = \frac{P_{\varepsilon\chi}}{P_\chi} < -\frac{h}{2}$.

Taking into account (2.8) also it is possible to express an integral $J_3$ through integrals $J_2$ and $J_1$:

$$J_3 = \frac{\sqrt{3}}{2}\sigma_s \left[ \frac{z}{2a}X^{\frac{1}{2}} - \frac{3b}{4a}\left( \frac{X^{\frac{1}{2}}}{a} - \frac{b}{2a}\int \frac{dz}{X^{\frac{1}{2}}} \right) - \frac{c}{2a}\int \frac{dz}{X^{\frac{1}{2}}} \right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

$$J_3 = \frac{\sqrt{3}}{2}\sigma_s \left[ \frac{z}{2a}X^{\frac{1}{2}} \right]_{-\frac{h}{2}}^{\frac{h}{2}} - \frac{3b}{4a}J_2 - \frac{c}{2a}J_1, \qquad (2.9)$$

$$X^{\frac{1}{2}} = \sqrt{c + bz + az^2},\ c = P_\varepsilon,\ b = -2P_{\varepsilon\chi},\ a = P_\chi.$$

Corresponding integrals according to (2.8)-(2.9):

$$J_1 = \frac{\sqrt{3}\sigma_s}{2} \cdot \frac{1}{\sqrt{P_\chi}} \times \ln \frac{2\sqrt{P_\chi} \cdot \sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} + hP_\chi - 2P_{\varepsilon\chi}}{2\sqrt{P_\chi} \cdot \sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} - hP_\chi - 2P_{\varepsilon\chi}}, \qquad (2.10)$$

$$J_2 = \frac{\sqrt{3}\sigma_s}{2} \cdot \frac{1}{P_\chi} \cdot \left( \sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} - \sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} \right) + \frac{P_{\varepsilon\chi}}{P_\chi}J_1, \qquad (2.11)$$

$$J_3 = \frac{\sqrt{3}\sigma_s}{2} \cdot \frac{1}{4P_\chi^2} \times \left( \left(hP_\chi + 6P_{\varepsilon\chi}\right)\sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} + \left(hP_\chi - 6P_{\varepsilon\chi}\right)\sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} \right) + \frac{3P_{\varepsilon\chi}^2 - P_\varepsilon P_\chi}{P_\chi^2}J_1. \qquad (2.12)$$

Taking into account (2.9) also it is possible to present an integral $J_3$ in a kind

$$J_3 = \frac{\sqrt{3}\sigma_s h}{2} \cdot \frac{1}{4P_\chi} \cdot \left( \sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} + \sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} \right) + \frac{3}{2}\frac{P_{\varepsilon\chi}}{P_\chi}J_2 - \frac{1}{2}\frac{P_\varepsilon}{P_\chi}J_1. \qquad (2.13)$$

At change of a sign $P_{\varepsilon\chi}$ integrals according to (2.10)–(2.13) $J_1 = J_1$, $J_2 = -J_2$, $J_3 = J_3$. If $P_\varepsilon \to 0$ $J_1 \to \infty$, $J_2 \to 0$, $J_3 \to \frac{\sqrt{3}\sigma_s}{2} \cdot \frac{h^2}{4\sqrt{P_\chi}}$.

Intensity of deformations (2.1) taking into account (2.4) becomes

$$e_i = \sqrt{e_{i0}^2 - \frac{z}{h}\left(e_{i1}^2 - e_{i2}^2\right) + \frac{z^2}{h^2}\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)}. \qquad (2.14)$$

According to (2.14) integrals in formulas (4.23')-(4.24') [9]:

$$J_1 = \sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dz}{X^{\frac{1}{2}}},\ J_2 = \sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{zdz}{X^{\frac{1}{2}}},\ J_3 = \sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z^2dz}{X^{\frac{1}{2}}},$$

$$X^{\frac{1}{2}} = \sqrt{c + bz + az^2},\ c = e_{i0}^2,\ b = -\frac{1}{h}\left(e_{i1}^2 - e_{i2}^2\right),$$

$$a = \frac{1}{h^2}\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right). \qquad (2.15)$$

Corresponding integrals according to (2.8)–(2.9) which can be received also substitution (2.4) in (2.10)–(2.13):

$$J_1 = \frac{\sigma_s h}{\sqrt{2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2}} \times \ln \frac{2e_{i2}\sqrt{2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2} + \left(e_{i1}^2 + 3e_{i2}^2 - 4e_{i0}^2\right)}{2e_{i1}\sqrt{2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2} - \left(3e_{i1}^2 + e_{i2}^2 - 4e_{i0}^2\right)}, \qquad (2.16)$$

$$J_2 = \frac{\sigma_s h^2\left(e_{i2} - e_{i1}\right)}{\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)} + \frac{h\left(e_{i1}^2 - e_{i2}^2\right)}{2\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)}J_1, \qquad (2.17)$$

$$J_3 = \frac{\sigma_s h^3\left(e_{i1} + e_{i2}\right)}{4\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)} - \frac{3\sigma_s h^3\left(e_{i1}^2 - e_{i2}^2\right)\left(e_{i1} - e_{i2}\right)}{4\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)^2} + \frac{h^2\left[3\left(e_{i1}^2 + e_{i2}^2\right)^2 - 4e_{i0}^2\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)\right]}{8\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)^2}J_1. \qquad (2.18)$$

Taking into account (2.9) also it is possible to present an integral $J_3$ in a kind

$$J_3=\frac{\sigma_s h^3\left(e_{i1}+e_{i2}\right)}{4\left(2e_{i1}^2+2e_{i2}^2-4e_{i0}^2\right)}+\frac{3h\left(e_{i1}^2-e_{i2}^2\right)J_2}{4\left(2e_{i1}^2+2e_{i2}^2-4e_{i0}^2\right)}-$$
$$-\frac{h^2e_{i0}^2}{2\left(2e_{i1}^2+2e_{i2}^2-4e_{i0}^2\right)}J_1. \tag{2.19}$$

Taking into account (2.5) formulas (2.16)-(2.19) become:

$$J_1=\frac{\sigma_s h}{e_{i1}\sqrt{2+2\lambda^2-4\mu^2}}\times$$
$$\times\ln\frac{2\lambda\sqrt{2+2\lambda^2-4\mu^2}+\left(1+3\lambda^2-4\mu^2\right)}{2\sqrt{2+2\lambda^2-4\mu^2}-\left(3+\lambda^2-4\mu^2\right)}, \tag{2.20}$$

$$J_2=\frac{\sigma_s h^2\left(\lambda-1\right)}{e_{i1}\left(2+2\lambda^2-4\mu^2\right)}+\frac{h\left(1-\lambda^2\right)}{2\left(2+2\lambda^2-4\mu^2\right)}J_1, \tag{2.21}$$

$$J_3=\frac{\sigma_s h^3\left(1+\lambda\right)}{4e_{i1}\left(2+2\lambda^2-4\mu^2\right)}-\frac{3\sigma_s h^3\left(1-\lambda^2\right)}{4e_{i1}\left(2+2\lambda^2-4\mu^2\right)^2}+$$
$$+\frac{h^2\left[3\left(1+\lambda^2\right)^2-4\mu^2\left(2+2\lambda^2-4\mu^2\right)\right]}{8\left(2+2\lambda^2-4\mu^2\right)^2}J_1, \tag{2.22}$$

$$J_3=\frac{\sigma_s h^3\left(1+\lambda\right)}{4e_{i1}\left(2+2\lambda^2-4\mu^2\right)}+$$
$$+\frac{3h\left(1-\lambda^2\right)}{4\left(2+2\lambda^2-4\mu^2\right)}J_2-\frac{h^2\mu^2}{2\left(2+2\lambda^2-4\mu^2\right)}J_1. \tag{2.23}$$

Formulas (4.44) taking into account (4.66)-(4.68) [9]

$$P_S=\frac{3}{4}\cdot\frac{\sigma_s^2h^2}{1}\left(n_1^2-n_1n_2+n_2^2+3n_{12}^2\right)=$$
$$=\frac{3}{4}\cdot\frac{\sigma_s^2h^2}{1}Q_n,$$
$$P_H=\frac{3}{4}\cdot\frac{\sigma_s^2h^4}{16}\left(m_1^2-m_1m_2+m_2^2+3m_{12}^2\right)=$$
$$=\frac{3}{4}\cdot\frac{\sigma_s^2h^4}{16}Q_m, \tag{2.24}$$
$$P_{SH}=\frac{3}{4}\cdot\frac{\sigma_s^2h^3}{4}\left(\begin{array}{l}n_1m_1+n_2m_2-\frac{1}{2}n_1m_2-\\-\frac{1}{2}n_2m_1+3n_{12}m_{12}\end{array}\right)=$$
$$=\frac{3}{4}\cdot\frac{\sigma_s^2h^3}{4}Q_{nm},$$

where

$$Q_n==\frac{4}{3}\cdot\frac{1}{\sigma_s^2h^2}P_S,\ Q_m=\frac{4}{3}\cdot\frac{16}{\sigma_s^2h^4}P_H,$$
$$Q_{nm}=\frac{4}{3}\cdot\frac{4}{\sigma_s^2h^3}P_{SH}. \tag{2.25}$$

From here with the account (4.45'), (4.45''), (4.45''') [9] we receive a required final relation:

$$Q_n=\frac{4}{3}\cdot\frac{1}{\sigma_s^2h^2}P_S=$$
$$=\frac{4}{3}\cdot\frac{1}{\sigma_s^2h^2}\left[J_1^2P_\varepsilon-2J_1J_2P_{\varepsilon\chi}+J_2^2P_\chi\right],$$
$$Q_{nm}=\frac{4}{3}\cdot\frac{4}{\sigma_s^2h^3}P_{SH}=$$
$$=\frac{4}{3}\cdot\frac{4}{\sigma_s^2h^3}\left[J_1J_2P_\varepsilon-\left(J_1J_3+J_2^2\right)P_{\varepsilon\chi}+J_2J_3P_\chi\right],$$
$$Q_m=\frac{4}{3}\cdot\frac{16}{\sigma_s^2h^4}P_H=$$
$$=\frac{4}{3}\cdot\frac{16}{\sigma_s^2h^4}\left[J_2^2P_\varepsilon-2J_2J_3P_{\varepsilon\chi}+J_3^2P_\chi\right]. \tag{2.26}$$

As in A.A. Ilyushin's theory $e_{i0}$ – the minimum value of intensity of deformations $e_i$ at $z=z_0$, and in offered model $e_{i0}$ – value of intensity of deformations $e_i$ at $z = 0$ also have different physical sense, we will designate these parametres as follows:

$$e_i\Big|_{z=z_0}=e_{i0,\min},\ \mu_{\min}=\frac{e_{i0,\min}}{e_{i1}},\ e_i\Big|_{z=0}=e_{i0},\ \mu=\frac{e_{i0}}{e_{i1}}.$$

The relationship between these parameters is obtained from (4.34) [9]

$$e_{i0,\min}^2=\frac{2}{\sqrt{3}}\sqrt{P_\varepsilon-\frac{P_{\varepsilon\chi}^2}{P_\chi}}, \tag{2.27}$$

Where $P_\varepsilon$, $P_{\varepsilon\chi}$, $P_\chi$ according to (2.4):

$$e_{i0,\min}^2=e_{i0}^2-\frac{\left(e_{i1}^2-e_{i2}^2\right)^2}{4\left(2e_{i1}^2+2e_{i2}^2-4e_{i0}^2\right)}, \tag{2.28}$$

$$\mu_{\min}^2=\mu^2-\frac{\left(1-\lambda^2\right)^2}{4\left(2+2\lambda^2-4\mu^2\right)}. \tag{2.29}$$

Deciding biquadratic the equations (2.28)-(2.29), we find

$$e_{i0}^2=\frac{1}{4}\left(\begin{array}{l}e_{i1}^2+e_{i2}^2+2e_{i0,\min}^2\mp\\\mp2\sqrt{e_{i1}^2-e_{i0,\min}^2\left(e_{i1}^2+e_{i2}^2\right)+e_{i0,\min}^4}\end{array}\right), \tag{2.30}$$

$$\mu^2=\frac{1}{4}\left(\begin{array}{l}1+\lambda^2+2\mu_{\min}^2\mp\\ \mp2\sqrt{\lambda^2-\mu_{\min}^2\left(1+\lambda^2\right)+\mu_{\min}^4}\end{array}\right). \quad (2.31)$$

In formulas (2.30)-(2.31) upper sign (–) concerns to a case of a dominating bending of a shell, and lower (+) to a case of a dominating stretching – compression. Analysing (2.29), (2.31), we find limits of change of parametres $\lambda$, $\mu_{min}$, $\mu$:

For a dominating bending of a shell:

$$\lambda=1,\ 0\le\mu_{\min}\le\lambda,\quad \mu=\mu_{\min};$$
$$\lambda<1,\ \mu_{\min}=0,\ \mu=\frac{1-\lambda}{2},\ 0\le\mu\le\frac{1}{2};$$
$$\lambda<1,\ \mu_{\min}=\lambda,\ \mu=\frac{\sqrt{1+3\lambda^2}}{2},\ \frac{1}{2}\le\mu<1;$$
$$\lambda=0,\ \mu_{\min}=0,\ \mu=\frac{1}{2}. \quad (2.32)$$

For the dominant extension – compression of the shell:

$$\lambda=1,\ 0\le\mu_{\min}\le\lambda,\ \ \mu=1;$$
$$\lambda<1,\ \mu_{\min}=0,\ \mu=\frac{1+\lambda}{2},\ \frac{1}{2}\le\mu\le1;$$
$$\lambda<1,\ \mu_{\min}=\lambda,\ \mu=\frac{\sqrt{1+3\lambda^2}}{2},\ \frac{1}{2}\le\mu<1;$$
$$\lambda=0,\ \mu_{\min}=0,\ \mu=\frac{1}{2}. \quad (2.33)$$

Another variant of the relation between the parameters $e_{i0,\min}$, $e_{i0}$, $\mu_{min}$, $\mu$ is obtained from (4.60) [9] and (2.4)

$$e_{i0}^2=\frac{1}{4}\left(\begin{array}{l}e_{i1}^2+e_{i2}^2+2e_{i0,\min}^2\mp\\ \mp2\sqrt{e_{i1}^2-e_{i0,\min}^2}\cdot\sqrt{e_{i2}^2-e_{i0,\min}^2}\end{array}\right), \quad (2.34)$$

$$\mu^2=\frac{1}{4}\left(1+\lambda^2+2\mu_{\min}^2\mp2\sqrt{1-\mu_{\min}^2}\cdot\sqrt{\lambda^2-\mu_{\min}^2}\right). \quad (2.35)$$

In formulas (2.34)-(2.35) upper sign (–) concerns to a case of a dominating bending of a shell, and lower (+) to a case of a dominating stretching – compression. Formulas (2.30), (2.34), (2.31), (2.35) are equivalent. Product of radicals in (2.34)–(2.35) is equal to a radical in (2.30)–(2.31). Limits of change of parametres are naturally identical. Deciding (2.34) and (2.35) rather $e_{i0,\min}$, $\mu_{min}$, we receive (2.28) and (2.29).

The right parts of system of the equations (2.26) are functions only two parametres $\lambda$, $\mu$, in three-dimensional space with variables $Q_n$, $Q_m$, $Q_{nm}$ they represent a surface $F(Q_n, Q_m, Q_{nm}) = 0$, and (2.26) is the parametric equation of this surface and coincides with (4.70') [9].

If to enter new functions by analogy with (4.62)-(4.65) [9] after enough bulky transformations of the right parts of the equations (2.26), relation (2.26) can be resulted in a kind (4.70') [9]

$$Q_n=Q_n\left[\begin{array}{l}\Delta_1(\lambda,\ \mu),\ \Delta(\lambda,\ \mu),\ \psi(\lambda,\ \mu),\\ \varphi(\lambda),\ \mu\end{array}\right],$$
$$Q_{nm}=Q_{nm}\left[\begin{array}{l}\Delta_1(\lambda,\ \mu),\ \Delta(\lambda,\ \mu),\ \psi(\lambda,\ \mu),\\ \varphi(\lambda),\ \mu\end{array}\right],$$
$$Q_m=Q_m\left[\begin{array}{l}\Delta_1(\lambda,\ \mu),\ \Delta(\lambda,\ \mu),\ \psi(\lambda,\ \mu),\\ \varphi(\lambda),\ \mu\end{array}\right],$$
$$\Delta_1^2=2+2\lambda^2-4\mu^2,\ \Delta=\frac{1-\lambda^2}{\Delta_1},$$
$$\psi=J_1\cdot\Delta_1,\ \varphi=\lambda-1. \quad (2.36)$$

It is possible to notice that function $\chi$ here does not enter, as and in (4.70') [9] it is not independent and is equal $\chi=\frac{(\lambda+1)\Delta_1}{2}-\frac{(\lambda-1)\Delta}{2}$.

Similar transformations are necessary in the absence of high-power computer facilities. Now in it there are no necessities and the right parts of the equations (2.26) are more convenient for calculating directly. Ratio (2.26) and (4.70') [9] are equivalent.

As well as in the work [9] we consider three special cases of a final relation.

***1. The momentless tension state*** occurs if the deformations of the fibers along the thickness of the shell are the same:

$$e_{i1}=e_{i2}=e_{i0}=e_{i0,\min},\ \lambda=\mu=\mu_{\min}=1.$$

In the formulas (2.31)-(2.35) it is necessary to take the lower sign (+). Expanding the uncertainties in the formulas (2.20)-(2.23) and (2.26), we obtain the Mizes condition (4.71)-(4.71') [9]

In formulas (2.31)-(2.35) it is necessary to take the lower sign (+). Opening uncertainty of formulas (2.20)-(2.23) and (2.26), we receive a condition of Mizes (4.71)-(4.71') [9]

$$Q_m=Q_{nm}=0,\ Q_n=n_1^2-n_1n_2+n_2^2+3n_{12}^2=1. \quad (2.37)$$

**2. *Purely moments the tension*** takes place in the absence of lengthening of a middle surface. Quadratic forms $P_\varepsilon = P_{\varepsilon\chi} = 0$.

As appears from (4.19) [9], intensity of deformations $e_i$ is even function $z$ and, according to (4.34) [9], (2.2) is had: $e_{i1} = e_{i2}$, $e_{i0} = e_{i0,\min} = 0$, $\lambda = 1$, $\mu = \mu_{\min} = 0$.

In formulas (2.31)-(2.35) it is necessary to take the upper sign (–). Opening uncertainty of formulas (2.20)-(2.23) and (2.26), we receive a condition (4.72)-(4.72') [9]. The final relation (4.70') [9] becomes:

$$Q_n = Q_{nm} = 0,\ Q_m = m_1^2 - m_1 m_2 + m_2^2 + 3m_{12}^2 = 1. \quad (2.38)$$

**3. *The elementary difficult tension*** of shells at $P_\chi \neq 0$, $P_\varepsilon \neq 0$ takes place, if the bilinear form

$$P_{\varepsilon\chi} = \varepsilon_1\chi_1 + \varepsilon_2\chi_2 + \frac{1}{2}\varepsilon_2\chi_1 + + \frac{1}{2}\varepsilon_1\chi_2 + \chi_{12}\varepsilon_{12} = 0.$$

Possible versions:

$$P_{\varepsilon\chi} = \chi_1\left(\varepsilon_1 + \frac{1}{2}\varepsilon_2\right) + \chi_2\left(\varepsilon_2 + \frac{1}{2}\varepsilon_1\right) + \chi_{12}\varepsilon_{12} = 0,$$

$$P_{\varepsilon\chi} = \varepsilon_1\left(\chi_1 + \frac{1}{2}\chi_2\right) + \varepsilon_2\left(\chi_2 + \frac{1}{2}\chi_1\right) + \chi_{12}\varepsilon_{12} = 0.$$

It can take place in cases (4.74) [9] and in addition:

$$a)\ \chi_1 \neq 0,\ \chi_{12} = \chi_2 = 0,\ \varepsilon_1 + \frac{1}{2}\varepsilon_2 = 0,$$

$$b)\ \chi_2 \neq 0,\ \chi_{12} = \chi_1 = 0,\ \varepsilon_2 + \frac{1}{2}\varepsilon_1 = 0,$$

$$c)\ \varepsilon_1 \neq 0,\ \varepsilon_{12} = \varepsilon_2 = 0,\ \chi_1 + \frac{1}{2}\chi_2 = 0,$$

$$d)\ \varepsilon_2 \neq 0,\ \varepsilon_{12} = \varepsilon_1 = 0,\ \chi_2 + \frac{1}{2}\chi_1 = 0,$$

$$e)\ \chi_1 = \chi_2,\ \varepsilon_1 = -\varepsilon_2,\ f)\ \chi_1 = -\chi_2,\ \varepsilon_1 = \varepsilon_2.$$

From (4.60) [9] – (2.4) it is had: $e_{i1} = e_{i2} > e_{i0} = e_{i0,\min}$, $\lambda = 1$, $\mu = \mu_{\min} < 1$, i.e. dominating bending strain. According to (2.6)

$$P_\varepsilon = \frac{3}{4}\mu^2 e_{i1}^2,\ hP_{\varepsilon\chi} = 0,\ \frac{h^2}{4}P_\chi = \frac{3}{4}\left(1-\mu^2\right)e_{i1}^2. \quad (2.39)$$

Corresponding integrals according to (2.20)-(2.23):

$$J_1 = \frac{\sigma_s h}{2e_{i1}\left(1-\mu^2\right)}\ln\frac{1+\sqrt{1-\mu^2}}{1-\sqrt{1-\mu^2}},\ J_2 = 0,$$

$$J_3 = \frac{\sigma_s h^2}{8e_{i1}\left(1-\mu^2\right)}\left(1-\mu^2 J_1\right). \quad (2.40)$$

The final relation (2.26) becomes:

$$Q_n = \frac{\mu^2}{4\left(1-\mu^2\right)}\ln^2\frac{1+\sqrt{1-\mu^2}}{1-\sqrt{1-\mu^2}},\ Q_{nm} = 0,$$

$$Q_m = \left(\frac{1}{\sqrt{1-\mu^2}} - \frac{\mu^2}{2\left(1-\mu^2\right)}\ln\frac{1+\sqrt{1-\mu^2}}{1-\sqrt{1-\mu^2}}\right)^2. \quad (2.41)$$

Considering identity (341.01 [42]

$$\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x} = \frac{1}{2a}\ln\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}},$$

the final relation (2.26) becomes (4.74) [9]:

$$Q_n = \frac{\mu^2}{1-\mu^2}\ln^2\frac{1+\sqrt{1-\mu^2}}{\mu},\ Q_{nm} = 0,$$

$$Q_m = \left(\frac{\mu^2}{1-\mu^2}\ln\frac{1+\sqrt{1-\mu^2}}{\mu} - \frac{1}{\sqrt{1-\mu^2}}\right)^2. \quad (2.41')$$

In table 2.1 shows the coordinates of points of a curve (2.42) and (4.74) [9] for the elementary difficult tension of a shell are presented.

**4. A difficult tension of shells** if the bilinear form $P_{\varepsilon\chi}$ submits to a relation $P_{\varepsilon\chi}^2 = P_\varepsilon \cdot P_\chi$.

In case of a dominating stretching of a shell at the lower sign (+) in (2.31) it is had:

$\lambda < 1$, $\mu_{\min} = 0$, $\mu = \frac{1+\lambda}{2}$. Substituting corresponding integrals in (2.26), we receive $Q_n = 1$, $Q_{nm} = Q_m = 0$, i.e. the line $\mu = 0$ degenerates in a point.

In case of a dominating bending the upper sign (–) in (2.31) it is received:

$$\lambda < 1,\ \mu_{\min} = 0,\ \mu = \frac{1-\lambda}{2}.$$

Substituting corresponding integrals in (2.26), we receive (4.79') [9], and excepting parametre $\lambda$ also (4.77), (4.79), (4.80) [9]

$$Q_n = \left(\frac{1-\lambda}{1+\lambda}\right)^2,\ Q_{nm} = -\frac{4\lambda(1-\lambda)}{(1+\lambda)^3},\ Q_m = \frac{16\lambda^2}{(1+\lambda)^4},$$

$$Q_{nm}^2 = Q_n \cdot Q_m,\ Q_m = \left(1-Q_n\right)^2,\ \left|Q_{nm}\right| = \left(1-Q_n\right)\sqrt{Q_n} \quad (2.42)$$

In table 2.2 coordinates of points of a surface (2.26) and (4.70) [9] on lines $\lambda = const$ for a dominating bending of a shell are presented $\lambda = const$.

In table 2.3 coordinates of points of a surface (2.26) and (4.70) [9] on lines $\lambda = const$ for a dominating stretching – compression are presented $\lambda = const$, in work [9] given table is not presented, is visible that gives small enough quantity of points in a vicinity $Q_n \to 1$ with ordinates $x = y = 0,3876$, $z = 0,3872$. Tables 2.4 and 2.5 is other form of representation of results of calculation. Table 2.4 corresponds to a dominating bending of a shell, table 2.5 – to a case to a dominating stretching – to compression.

In relation (2.26) integrals (4.25) [9] are calculated under unified (unequivocal) formulas. The account of a dominating bending of a shell and a dominating stretching – compression is executed at level of communication of parametres $\mu$ and $\mu_{min}$.

Let us show that a finite relation can be obtained using the parameters of A.A. Ilyushin and calculating the integrals (4.25) [9] along the coordinate $z$.

Quadratic forms according to (4.60) [9]:

$$hP_{\varepsilon\chi} = \frac{3}{8}\left(e_{i1}^2 - e_{i2}^2\right),$$

$$P_{\varepsilon} = \frac{3}{16}\left[\left(\sqrt{e_{i1}^2 - e_{i0}^2} \mp \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2 + 4e_{i0}^2\right] =$$

$$= \frac{3}{16}\left[2\left(e_{i1}^2 + e_{i2}^2\right) - \left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2\right],$$

$$\frac{h^2}{4}P_{\chi} = \frac{3}{16}\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2. \qquad (2.43)$$

Substituting (2.43) into (2.10)–(2.13), we obtain the integrals $J_1$, $J_2$, $J_3$:

$$J_1 = \frac{\sigma_s h}{\left|\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right|} \times \ln \frac{e_{i2}\left|\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right| + \sqrt{e_{i2}^2 - e_{i0}^2}\cdot\left(\sqrt{e_{i2}^2 - e_{i0}^2} \pm \sqrt{e_{i1}^2 - e_{i0}^2}\right)}{e_{i1}\left|\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right| - \sqrt{e_{i1}^2 - e_{i0}^2}\cdot\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)}, \qquad (2.44)$$

$$J_2 = \frac{\sigma_s h^2\left(e_{i2} - e_{i1}\right)}{\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2} + \frac{h\left(e_{i1}^2 - e_{i2}^2\right)}{2\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2} J_1, \qquad (2.45)$$

$$J_3 = \frac{\sigma_s h^3\left(e_{i2} + e_{i1}\right)}{4\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2} + \frac{3h\left(e_{i1}^2 - e_{i2}^2\right)}{4\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2} J_2 - \frac{h^2\left[\left(\sqrt{e_{i1}^2 - e_{i0}^2} \mp \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2 + 4e_{i0}^2\right]}{8\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2} J_1. \qquad (2.46)$$

Taking into account the introduction of two basic parameters $\lambda$ and $\mu$ according to (4.61) [9] $\lambda = \frac{e_{i2}}{e_{i1}}$, $\mu = \frac{e_{i0}}{e_{i1}}$, the relations (2.43) take the form

$$hP_{\varepsilon\chi} = \frac{3}{8e_{i1}^2}\left(1 - \lambda^2\right),$$

$$P_{\varepsilon} = \frac{3}{16e_{i1}^2}\left[\left(\sqrt{1-\mu^2} \mp \sqrt{\lambda^2 - \mu^2}\right)^2 + 4\mu^2\right],$$

$$\frac{h^2}{4}P_{\chi} = \frac{3}{16e_{i1}^2}\left(\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right)^2, \qquad (2.47)$$

and the integrals $J_1$, $J_2$, $J_3$ are expressed in terms of the basic parameters $\lambda$ and $\mu$:

$$J_1 = \frac{\sigma_s h}{e_{i1}\left|\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right|} \times \ln \frac{\lambda\left|\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right| + \sqrt{\lambda^2 - \mu^2}\cdot\left(\sqrt{\lambda^2 - \mu^2} \pm \sqrt{1-\mu^2}\right)}{\left|\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right| - \sqrt{1-\mu^2}\cdot\left(\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right)}, \qquad (2.48)$$

$$J_2 = \frac{\sigma_s h^2\left(\lambda - 1\right)}{e_{i1}\left(\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right)^2} + \frac{h^2\left(1 - \lambda^2\right)}{\left(\sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2}\right)^2} J_1, \qquad (2.49)$$

***Table 2.1. Coordinates curve $Q_n$, $Q_m$ (the expanded version of table 4 [9]).***

| μ | | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | |
| 0.00 | $Q_n$ | 0.0000 | 0.0905 | 0.2190 | 0.3473 | 0.4676 | 0.5781 | 0.6789 | 0.7706 | 0.8541 | 0.9303 |
| | $Q_m$ | 1.0000 | 0.9502 | 0.8558 | 0.7447 | 0.6283 | 0.5122 | 0.3995 | 0.2914 | 0.1887 | 0.0916 |
| 0.01 | $Q_n$ | 0.0028 | 0.1028 | 0.2320 | 0.3597 | 0.4791 | 0.5886 | 0.6885 | 0.7793 | 0.8621 | 0.9376 |
| | $Q_m$ | 0.9990 | 0.9421 | 0.8452 | 0.7332 | 0.6166 | 0.5008 | 0.3884 | 0.2809 | 0.1787 | 0.0822 |
| 0.02 | $Q_n$ | 0.0085 | 0.1153 | 0.2450 | 0.3721 | 0.4905 | 0.5990 | 0.6980 | 0.7880 | 0.8699 | 0.9448 |
| | $Q_m$ | 0.9967 | 0.9336 | 0.8344 | 0.7216 | 0.6049 | 0.4894 | 0.3774 | 0.2704 | 0.1688 | 0.0728 |
| 0.03 | $Q_n$ | 0.0159 | 0.1280 | 0.2580 | 0.3843 | 0.5018 | 0.6094 | 0.7073 | 0.7965 | 0.8777 | 0.9519 |
| | $Q_m$ | 0.9933 | 0.9248 | 0.8236 | 0.7100 | 0.5933 | 0.4780 | 0.3665 | 0.2600 | 0.1590 | 0.0635 |
| 0.04 | $Q_n$ | 0.0245 | 0.1409 | 0.2710 | 0.3965 | 0.5130 | 0.6196 | 0.7166 | 0.8050 | 0.8854 | 0.9589 |
| | $Q_m$ | 0.9891 | 0.9156 | 0.8125 | 0.6984 | 0.5816 | 0.4666 | 0.3556 | 0.2497 | 0.1492 | 0.0543 |
| 0.05 | $Q_n$ | 0.0341 | 0.1538 | 0.2839 | 0.4086 | 0.5241 | 0.6297 | 0.7258 | 0.8134 | 0.8931 | 0.9659 |
| | $Q_m$ | 0.9841 | 0.9062 | 0.8014 | 0.6867 | 0.5700 | 0.4553 | 0.3448 | 0.2394 | 0.1395 | 0.0451 |
| 0.06 | $Q_n$ | 0.0444 | 0.1667 | 0.2967 | 0.4206 | 0.5351 | 0.6397 | 0.7350 | 0.8217 | 0.9007 | 0.9729 |
| | $Q_m$ | 0.9784 | 0.8966 | 0.7902 | 0.6751 | 0.5584 | 0.4441 | 0.3340 | 0.2291 | 0.1298 | 0.0360 |
| 0.07 | $Q_n$ | 0.0553 | 0.1798 | 0.3094 | 0.4325 | 0.5460 | 0.6497 | 0.7440 | 0.8299 | 0.9082 | 0.9797 |
| | $Q_m$ | 0.9721 | 0.8867 | 0.7790 | 0.6634 | 0.5468 | 0.4328 | 0.3233 | 0.2189 | 0.1201 | 0.0269 |
| 0.08 | $Q_m$ | 0.0667 | 0.1928 | 0.3221 | 0.4443 | 0.5568 | 0.6595 | 0.7530 | 0.8380 | 0.9156 | 0.9865 |
| | $Q_m$ | 0.9653 | 0.8766 | 0.7676 | 0.6517 | 0.5353 | 0.4217 | 0.3126 | 0.2088 | 0.1106 | 0.0179 |
| 0.09 | $Q_n$ | 0.0784 | 0.2059 | 0.3347 | 0.4560 | 0.5675 | 0.6693 | 0.7618 | 0.8461 | 0.9230 | 0.9933 |
| | $Q_m$ | 0.9580 | 0.8663 | 0.7562 | 0.6400 | 0.5237 | 0.4105 | 0.3020 | 0.1987 | 0.1011 | 0.0089 |
| 1.00 | $Q_n$ | | | | | | | | | | 1.0000 |
| | $Q_m$ | | | | | | | | | | 0.0000 |

$$J_3 = \frac{\sigma_s h^3 (\lambda + 1)}{4 e_{i1} \left( \sqrt{1-\mu^2} \pm \sqrt{\lambda^2 - \mu^2} \right)^2} + \frac{3h(1-\lambda^2)}{4\left( \sqrt{1-\mu^2} \pm \sqrt{\lambda^2-\mu^2} \right)^2} J_2 - \frac{h^2 \left[ \left( \sqrt{1-\mu^2} \mp \sqrt{\lambda^2-\mu^2} \right)^2 + 4\mu^2 \right]}{8\left( \sqrt{1-\mu^2} \pm \sqrt{\lambda^2-\mu^2} \right)^2} J_1. \quad (2.50)$$

In case of dominating bending strains of the formula (2.44)-(2.46) become:

$$J_1 = \frac{\sigma_s h}{\left| \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right|} \times \ln \frac{\left( e_{i1} + \sqrt{e_{i1}^2 - e_{i0}^2} \right)\left( e_{i2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)}{e_{i0}^2}, \quad (2.51)$$

$$J_2 = \frac{\sigma_s h^2 (e_{i2} - e_{i1})}{\left( \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} + \frac{h(e_{i1}^2 - e_{i2}^2)}{2\left( \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} J_1, \quad (2.52)$$

$$J_3 = \frac{\sigma_s h^3 (e_{i2} + e_{i1})}{4\left( \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} + \frac{3h(e_{i1}^2 - e_{i2}^2)}{4\left( \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} J_2 - \frac{h^2 \left[ \left( \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2 + 4e_{i0}^2 \right]}{8\left( \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} J_1. \quad (2.53)$$

In case of dominating lengthening of a middle surface from formulas (2.44)-(2.46) it is found:

$$J_1 = \frac{\sigma_s h}{\left| \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right|} \left| \ln \frac{\left( e_{i2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)}{\left( e_{i1} + \sqrt{e_{i1}^2 - e_{i0}^2} \right)} \right|, \quad (2.54)$$

$$J_2 = \frac{\sigma_s h^2 (e_{i2} - e_{i1})}{\left( \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} + \frac{h(e_{i1}^2 - e_{i2}^2)}{2\left( \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} J_1, \quad (2.55)$$

$$J_3 = \frac{\sigma_s h^3 (e_{i2} + e_{i1})}{4\left( \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} + \frac{3h(e_{i1}^2 - e_{i2}^2)}{4\left( \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} J_2 - \frac{h^2 \left[ \left( \sqrt{e_{i1}^2 - e_{i0}^2} + \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2 + 4e_{i0}^2 \right]}{8\left( \sqrt{e_{i1}^2 - e_{i0}^2} - \sqrt{e_{i2}^2 - e_{i0}^2} \right)^2} J_1. \quad (2.56)$$

Taking into account introduction of two key parametres λ and μ according to (4.61) [9] in case of dominating bending strains of the formula (2.51)-(2.53) become:

$$J_1 = \frac{\sigma_s h}{\left| \sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2} \right|} \times \ln \frac{\left( 1 + \sqrt{1-\mu^2} \right)\left( \lambda + \sqrt{\lambda^2 - \mu^2} \right)}{\mu^2}, \quad (2.57)$$

$$J_2 = \frac{\sigma_s h^2 (\lambda - 1)}{\left( \sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2} \right)^2} + \frac{h(1-\lambda^2)}{2\left( \sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2} \right)^2} J_1, \quad (2.58)$$

$$J_3 = \frac{\sigma_s h^3 (\lambda + 1)}{4\left( \sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2} \right)^2} + \frac{3h(1-\lambda^2)}{4\left( \sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2} \right)^2} J_2 - \frac{h^2 \left[ \left( \sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2} \right)^2 + 4\mu^2 \right]}{8\left( \sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2} \right)^2} J_1. \quad (2.59)$$

In case of dominating lengthening of a middle surface from formulas (2.54)-(2.56) it is found:

$$J_1 = \frac{\sigma_s h}{\left| \sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2} \right|} \left| \ln \frac{\left( \lambda + \sqrt{\lambda^2 - \mu^2} \right)}{\left( 1 + \sqrt{1-\mu^2} \right)} \right| \quad (2.60)$$

$$J_2 = \frac{\sigma_s h^2 (\lambda - 1)}{\left( \sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2} \right)^2} + \frac{h(1-\lambda^2)}{2\left( \sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2} \right)^2} J_1, \quad (2.61)$$

$$J_3 = \frac{\sigma_s h^3 (\lambda +1)}{4\left(\sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2}\right)^2} + \\ + \frac{3h(1-\lambda^2)}{4\left(\sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2}\right)^2} J_2 - \\ - \frac{h^2\left[\left(\sqrt{1-\mu^2} + \sqrt{\lambda^2 - \mu^2}\right)^2 + 4\mu^2\right]}{8\left(\sqrt{1-\mu^2} - \sqrt{\lambda^2 - \mu^2}\right)^2} J_1. \quad (2.62)$$

Intensity of deformations (2.1), taking into account (2.43) it is possible to present in a kind:

$$e_i = \sqrt{\begin{array}{l}\frac{1}{4}\left[\left(\sqrt{e_{i1}^2 - e_{i0}^2} \mp \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2 + 4e_{i0}^2\right] - \\ -\frac{z}{h}\left(e_{i1}^2 - e_{i2}^2\right) + \frac{z^2}{h^2}\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2\end{array}}. \quad (2.63)$$

According to (2.63) integrals in formulas (4.25) [9]:

$$J_1 = \sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{dz}{X^{\frac{1}{2}}},\ J_2 = \sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{zdz}{X^{\frac{1}{2}}},\ J_3 = \sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z^2dz}{X^{\frac{1}{2}}},$$

$$X^{\frac{1}{2}} = \sqrt{c + bz + az^2},$$

$$c = \frac{1}{4}\left[\left(\sqrt{e_{i1}^2 - e_{i0}^2} \mp \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2 + 4e_{i0}^2\right],$$

$$b = -\frac{1}{h}\left(e_{i1}^2 - e_{i2}^2\right),\ a = \frac{1}{h^2}\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2. \quad (2.64)$$

Considering (2.8)-(2.9), transformations become less bulky and to receive (2.44)-(2.46) it is possible much more fast.

The relations (2.44)-(2.46) are equivalent to (4.38), (4.59), (4.60) [9]. This can be seen if (4.38) [9] leads to the form

$$J_1 = \frac{\sqrt{3}}{2P_\chi^{\frac{1}{2}}} B,\ J_2 = \frac{P_{\varepsilon\chi}}{P_\chi} J_1 + \frac{3}{4P_\chi} A, \\ J_3 = \frac{3\sqrt{3}}{8P_\chi^{\frac{3}{2}}} C + \frac{2P_{\varepsilon\chi}}{P_\chi} J_2 - \frac{P_{\varepsilon\chi}^2}{P_\chi^2} J_1 \quad (2.65)$$

and to consider identities

$$hP_{\varepsilon\chi} = \frac{3}{8}\left(e_{i1}^2 - e_{i2}^2\right),$$

$$P_\varepsilon = \frac{3}{16}\left[\left(\sqrt{e_{i1}^2 - e_{i0}^2} \mp \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2 + 4e_{i0}^2\right] = \\ = \frac{3}{16}\left[2\left(e_{i1}^2 + e_{i2}^2\right) - \left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2\right],$$

$$\frac{h^2}{4} P_\chi = \frac{3}{16}\left(\sqrt{e_{i1}^2 - e_{i0}^2} \pm \sqrt{e_{i2}^2 - e_{i0}^2}\right)^2. \quad (2.66)$$

The relations (4.45) [9] – (2.26) can be given a different form if we introduce the new integral according to (4.28) [9]

$$J_0 = J_1 P_\varepsilon - 2J_2 P_{\varepsilon\chi} + J_3 P_\chi = \frac{3}{4}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s e_i dz = \\ = \frac{\sqrt{3}}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_s \sqrt{P_\varepsilon - 2zP_{\varepsilon\chi} + z^2 P_\chi}\, dz,$$

$$J_0 = \frac{\sqrt{3}}{2}\sigma_s \int_{-\frac{h}{2}}^{\frac{h}{2}} X^{\frac{1}{2}} dz,\ X^{\frac{1}{2}} = \sqrt{c + bz + az^2},$$

$$c = P_\varepsilon,\ b = -2P_{\varepsilon\chi},\ a = P_\chi. \quad (2.67)$$

This integral tabular. According to the formula 380.201 [42]

$$J_0 = \frac{\sqrt{3}}{2}\sigma_s \left[\left(\frac{2az + b}{4a}\right) X^{\frac{1}{2}} + \left(\frac{4ac - b^2}{8a}\right)\int \frac{dz}{X^{\frac{1}{2}}}\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

$$X^{\frac{1}{2}} = \sqrt{c + bz + az^2},\ c = P_\varepsilon,\ b = -2P_{\varepsilon\chi},\ a = P_\chi. \quad (2.68)$$

From here follows

$$J_0 = \frac{\sqrt{3}\sigma_s}{2} \cdot \frac{1}{4P_\chi} \times \\ \times \left(\begin{array}{l}\left(hP_\chi - 2P_{\varepsilon\chi}\right)\sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi} + \\ + \left(hP_\chi + 2P_{\varepsilon\chi}\right)\sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4}P_\chi}\end{array}\right) + \\ + \frac{P_\varepsilon P_\chi - P_{\varepsilon\chi}^2}{2P_\chi} J_1. \quad (2.69)$$

Considering (2.8), (2.68) it is possible to express an integral through integrals $J_1$ and $J_2$:

$$J_0 = \frac{\sqrt{3}}{2}\sigma_s \left[\begin{array}{l}\frac{z}{2} \cdot X^{\frac{1}{2}} + \frac{b}{4}\left(\frac{X^{\frac{1}{2}}}{a} - \frac{b}{2a}\right)\int \frac{dz}{X^{\frac{1}{2}}} + \\ + \frac{c}{2}\int \frac{dz}{X^{\frac{1}{2}}}\end{array}\right]_{-\frac{h}{2}}^{\frac{h}{2}} = \\ = \frac{\sqrt{3}}{2}\sigma_s \left[\frac{z}{2} \cdot X^{\frac{1}{2}}\right]_{-\frac{h}{2}}^{\frac{h}{2}} + \frac{b}{4} J_2 + \frac{c}{2} J_1,$$

$$X^{\frac{1}{2}} = \sqrt{c + bz + az^2},\ c = P_\varepsilon,\ b = -2P_{\varepsilon\chi},\ a = P_\chi. \quad (2.70)$$

Then (2.69) becomes (2.71)

$$J_0 = \frac{\sqrt{3}\sigma_s}{2} \cdot \frac{1}{4} \times$$
$$\times \left( \sqrt{P_\varepsilon - hP_{\varepsilon\chi} + \frac{h^2}{4} P_\chi} + \sqrt{P_\varepsilon + hP_{\varepsilon\chi} + \frac{h^2}{4} P_\chi} \right) -$$
$$-\frac{P_{\varepsilon\chi}}{2} J_2 + \frac{P_\varepsilon}{2} J_1 \tag{2.71}$$

Integral (2.69) taking into account (2.4)

$$J_0 = \frac{\sigma_s h \left[ \begin{array}{l} \left(e_{i1}^2 + 3e_{i2}^2 - 4e_{i0}^2\right) e_{i2} + \\ + \left(3e_{i1}^2 + e_{i2}^2 - 4e_{i0}^2\right) e_{i1} \end{array} \right]}{4\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)} +$$
$$+ \frac{1}{2} \left[ e_{i0}^2 - \frac{\left(e_{i1}^2 - e_{i2}^2\right)^2}{4\left(2e_{i1}^2 + 2e_{i2}^2 - 4e_{i0}^2\right)} \right] J_1. \tag{2.72}$$

According to (2.5) formula (2.72) for an integral $J_0$ becomes

$$J_0 = \frac{\sigma_s h\left[\left(1+3\lambda^2-4\mu^2\right)\lambda \ + \left(3+\lambda^2-4\mu^2\right)\right]}{4e_{i1}\left(2+2\lambda^2-4\mu^2\right)} +$$
$$+ \frac{1}{2}\left[\mu^2 - \frac{\left(1-\lambda^2\right)^2}{4\left(2+2\lambda^2-4\mu^2\right)}\right] J_1. \tag{2.73}$$

The final relation (4.45) [9] – (2.26) taking into account (2.67) takes the form:

$$P_S = J_1 J_0 - (J_1 J_3 - J_2^2) P_\chi$$
$$P_H = J_3 J_0 - (J_1 J_3 - J_2^2) P_\varepsilon \tag{2.74}$$
$$P_{SH} = J_2 J_0 - (J_1 J_3 - J_2^2) P_{\varepsilon\chi} .$$

The relations (2.74), (2.26) and (4.70') [9] are equivalent.

## 2.2. Approximate dependencies of the final relation

The integrals $J_1, J_2, J_3, J_0$ can be found by the Simpson formula, performing integration within each half of the section, since the intensity of deformations $e_i$ function can lose monotonicity at $z = 0$. According to (2.14–2.15), (2.67), the approximate values of the integrals:

$$J_1 = \frac{\sigma_s h}{12} \left( \begin{array}{l} \frac{1}{e_{i1}} + \frac{1}{e_{i2}} + \frac{2}{e_{i0}} + \frac{16}{\sqrt{12e_{i0}^2 + 6e_{i1}^2 - 2e_{i2}^2}} + \\ + \frac{16}{\sqrt{12e_{i0}^2 + 6e_{i2}^2 - 2e_{i1}^2}} \end{array} \right),$$

$$J_2 = \frac{\sigma_s h^2}{24} \left( \begin{array}{l} -\frac{1}{e_{i1}} + \frac{1}{e_{i2}} - \frac{8}{\sqrt{12e_{i0}^2 + 6e_{i1}^2 - 2e_{i2}^2}} + \\ + \frac{8}{\sqrt{12e_{i0}^2 + 6e_{i2}^2 - 2e_{i1}^2}} \end{array} \right),$$

$$J_3 = \frac{\sigma_s h^3}{48} \left( \begin{array}{l} \frac{1}{e_{i1}} + \frac{1}{e_{i2}} + \frac{4}{\sqrt{12e_{i0}^2 + 6e_{i1}^2 - 2e_{i2}^2}} + \\ + \frac{4}{\sqrt{12e_{i0}^2 + 6e_{i2}^2 - 2e_{i1}^2}} \end{array} \right), \tag{2.75}$$

$$J_0 = \frac{3}{4} \cdot \frac{\sigma_s h}{12} \times$$
$$\times \left( \begin{array}{l} e_{i1} + e_{i2} + 2e_{i0} + \\ + \sqrt{12e_{i0}^2 + 6e_{i1}^2 - 2e_{i2}^2} + \sqrt{12e_{i0}^2 + 6e_{i2}^2 - 2e_{i1}^2} \end{array} \right). \tag{2.76}$$

Taking into account (2.5) formulas (2.75)-(2.76) become:

$$J_1 = \frac{\sigma_s h}{12 e_{i1}} \left( \begin{array}{l} 1 + \frac{1}{\lambda} + \frac{2}{\mu} + \frac{16}{\sqrt{12\mu^2 + 6 - 2\lambda^2}} + \\ + \frac{16}{\sqrt{12\mu^2 + 6\lambda^2 - 2}} \end{array} \right),$$

$$J_2 = \frac{\sigma_s h^2}{24 e_{i1}} \left( \begin{array}{l} -1 + \frac{1}{\lambda} - \frac{8}{\sqrt{12\mu^2 + 6 - 2\lambda^2}} + \\ + \frac{8}{\sqrt{12\mu^2 + 6\lambda^2 - 2}} \end{array} \right),$$

$$J_3 = \frac{\sigma_s h^3}{48 e_{i1}} \left( \begin{array}{l} 1 + \frac{1}{\lambda} + \frac{4}{\sqrt{12\mu^2 + 6 - 2\lambda^2}} + \\ + \frac{4}{\sqrt{12\mu^2 + 6\lambda^2 - 2}} \end{array} \right), \tag{2.77}$$

$$J_0 = \frac{3}{4} \cdot \frac{\sigma_s h}{12 e_{i1}} \left( \begin{array}{l} 1 + \lambda + 2\mu + \sqrt{12\mu^2 + 6 - 2\lambda^2} + \\ + \sqrt{12\mu^2 + 6\lambda^2 - 2} \end{array} \right). \tag{2.78}$$

Believing that within each half of section intensity of deformations $e_i$ changes under the linear law

$$e_i = e_{i0} + \frac{2z}{h}\left(e_{i2} - e_{i0}\right), \quad 0 \le z \le \frac{h}{2},$$
$$e_i = e_{i0} - \frac{2z}{h}\left(e_{i1} - e_{i0}\right), \quad -\frac{h}{2} \le z \le 0$$

According to formulas (90.1, 91.1, 92.1) [42]

$$J_1 = \frac{\sigma_s}{b}\left[\ln|a+bz|\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

$$J_2 = \frac{\sigma_s}{b^2}\left[(a+bz) - a\ln|a+bz|\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

$$J_3 = \frac{\sigma_s}{b^3}\left[\frac{(a+bz)^2}{2} - 2a(a+bz) + a^2\ln|a+bz|\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

$$a = e_{i0},\ b = \frac{1}{h}(e_{i2} - e_{i0}),\ 0 \le z \le \frac{h}{2},$$

$$b = -\frac{1}{h}(e_{i1} - e_{i0}),\ -\frac{h}{2} \le z \le 0. \tag{2.79}$$

From here follows:

$$J_1 = \frac{\sigma_s h}{2(e_{i2} - e_{i0})}\ln\frac{e_{i2}}{e_{i0}} + \frac{\sigma_s h}{2(e_{i1} - e_{i0})}\ln\frac{e_{i1}}{e_{i0}},$$

$$J_2 = -\frac{\sigma_s h^2}{4(e_{i1} - e_{i0})^2}\left[(e_{i1} - e_{i0}) - e_{i0}\ln\frac{e_{i1}}{e_{i0}}\right] +$$

$$+\frac{\sigma_s h^2}{4(e_{i2} - e_{i0})^2}\left[(e_{i2} - e_{i0}) - e_{i0}\ln\frac{e_{i2}}{e_{i0}}\right],$$

$$J_3 = \frac{\sigma_s h^3}{16(e_{i1} - e_{i0})^3}\left[\begin{matrix}(3e_{i0}^2 + e_{i1}^2 - 4e_{i0}e_{i1}) + \\ +2e_{i0}^2\ln\frac{e_{i1}}{e_{i0}}\end{matrix}\right] +$$

$$+\frac{\sigma_s h^3}{16(e_{i2} - e_{i0})^3}\left[(3e_{i0}^2 + e_{i2}^2 - 4e_{i0}e_{i2}) + 2e_{i0}^2\ln\frac{e_{i2}}{e_{i0}}\right], \tag{2.80}$$

$$J_0 = \frac{3}{4}\cdot\frac{\sigma_s h}{4}(e_{i1} + e_{i2} + 2e_{i0}). \tag{2.81}$$

Taking into account (2.5) formulas (2.80)-(2.81) become:

$$J_1 = \frac{\sigma_s h}{2(\lambda - \mu)e_{i1}}\ln\frac{\lambda}{\mu} + \frac{\sigma_s h}{2(1-\mu)e_{i1}}\ln\frac{1}{\mu},$$

$$J_2 = -\frac{\sigma_s h^2}{4(1-\mu)^2 e_{i1}}\left[(1-\mu) - \mu\ln\frac{1}{\mu}\right] +$$

$$+\frac{\sigma_s h^2}{4(\lambda-\mu)^2 e_{i1}}\left[(\lambda-\mu) - \mu\ln\frac{\lambda}{\mu}\right],$$

$$J_3 = \frac{\sigma_s h^3}{16(1-\mu)^3 e_{i1}}\left[(3\mu^2 + 1 - 4\mu) + 2\mu^2\ln\frac{1}{\mu}\right] +$$

$$+\frac{\sigma_s h^3}{16(\lambda-\mu)^3 e_{i1}}\left[(3\mu^2 + \lambda^2 - 4\mu\lambda) + 2\mu^2\ln\frac{\lambda}{\mu}\right], \tag{2.82}$$

$$J_0 = \frac{3}{4}\cdot\frac{\sigma_s h}{4e_{i1}}(1 + \lambda + 2\mu). \tag{2.83}$$

Integrals $J_1$, $J_2$, $J_3$, $J_0$ (2.80) and (2.83) also can be found under Simpson's formula, executing integration within each half of section

$$J_1 = \frac{\sigma_s h}{12}\left(\frac{1}{e_{i1}} + \frac{1}{e_{i2}} + \frac{2}{e_{i0}} + \frac{8}{(e_{i1} + e_{i0})} + \frac{8}{(e_{i2} + e_{i0})}\right),$$

$$J_2 = \frac{\sigma_s h^2}{24}\left(-\frac{1}{e_{i1}} + \frac{1}{e_{i2}} - \frac{4}{(e_{i1} + e_{i0})} + \frac{4}{(e_{i2} + e_{i0})}\right),$$

$$J_3 = \frac{\sigma_s h^3}{48}\left(\frac{1}{e_{i1}} + \frac{1}{e_{i2}} + \frac{2}{(e_{i1} + e_{i0})} + \frac{2}{(e_{i2} + e_{i0})}\right), \tag{2.84}$$

$$J_0 = \frac{3}{4}\cdot\frac{\sigma_s h}{4}(e_{i1} + e_{i2} + 2e_{i0}). \tag{2.85}$$

$$J_1 = \frac{\sigma_s h}{12e_{i1}}\left(1 + \frac{1}{\lambda} + \frac{2}{\mu} + \frac{8}{1+\mu} + \frac{8}{\lambda+\mu}\right),$$

$$J_2 = \frac{\sigma_s h^2}{24e_{i1}}\left(-1 + \frac{1}{\lambda} - \frac{4}{1+\mu} + \frac{4}{\lambda+\mu}\right),$$

$$J_3 = \frac{\sigma_s h^3}{48e_{i1}}\left(1 + \frac{1}{\lambda} + \frac{2}{1+\mu} + \frac{2}{\lambda+\mu}\right),$$

$$J_0 = \frac{3}{4}\cdot\frac{\sigma_s h}{4e_{i1}}(1 + \lambda + 2\mu). \tag{2.86}$$

On the basis regression the analysis of a curve (2.41) (the minimum line $Q_{nm}$, table 2.1) are received versions of its approximation by polynoms of the second, third and fourth degree and its first derivative is found:

Polynom of the second degree.

$$Q_m = 1.0099235 - 0.642635\cdot Q_n - 0.3718551\cdot Q_n^2,$$

$$y = 1.0099235 - 0.642635x - 0.3718551x^2,$$

$$\frac{\partial y}{\partial x} = -0.642635 - 2\cdot 0.3718551x,$$

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = -0.6426,\ \left.\frac{\partial y}{\partial x}\right|_{x=1} = -1.3864.$$

Polynom of the third degree.

$$Q_m = 1.0037431 - 0.5575663 \cdot Q_n - 0.586892 \cdot Q_n^2 + $$
$$+ 0.1423386 \cdot Q_n^3,$$
$$y = 1.0037431 - 0.5575663x - 0.586892x^2 +$$
$$+ 0.1423386x^3,$$
$$\frac{\partial y}{\partial x} = -0.5575663 - 2 \cdot 0.586892x + 3 \cdot 0.1423386x^2,$$
$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = -0.5576, \quad \left.\frac{\partial y}{\partial x}\right|_{x=1} = -1.3043.$$

Polynom of the fourth degree.

$$Q_m = 1.0019337 - 0.5128967 \cdot Q_n -$$
$$-0.7948953 \cdot Q_n^2 + 0.4669156 \cdot Q_n^3 -$$
$$-0.1613785 \cdot Q_n^4,$$
$$y = 1.0019337 - 0.5128967x - 0.7948953x^2 +$$
$$+0.4669156x^3 - 0.1613785x^4,$$
$$\frac{\partial y}{\partial x} = -0.5128967 - 2 \cdot 0.7948953x +$$
$$+3 \cdot 0.4669156x^2 - 4 \cdot 0.1613785x^3,$$
$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = -0.5129, \quad \left.\frac{\partial y}{\partial x}\right|_{x=1} = -1.3475.$$

In fig. 2.1-2.3 the curve (2.41) (table 2.1) (the minimum line $Q_{nm}$) is presented, the variants of its approximation by polynomials of the second, third and fourth degree (the lines merge) and its first derivative on the basis of regression analysis. As you can see from the graphs, a polynomial of the second degree is sufficient.

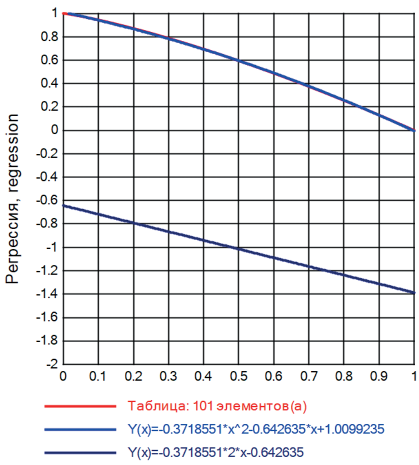


*Figure 2.1. Curve (2.41) (table 2.1, a minimum line $Q_{nm}$), version of approximation by a polynom of the second degree and its first derivative on the basis regression the analysis.*

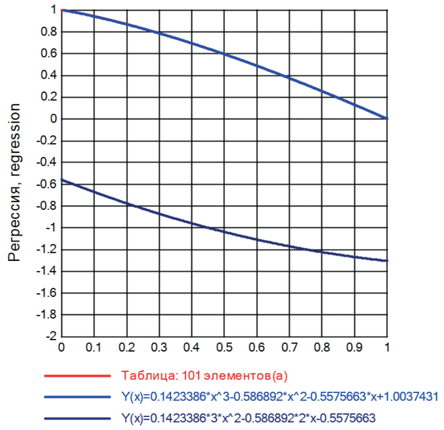


*Figure 2.2. Curve (2.41) (table 2.1, a minimum line $Q_{nm}$), version of approximation by a polynom of the third degree and its first derivative on the basis regression the analysis.*

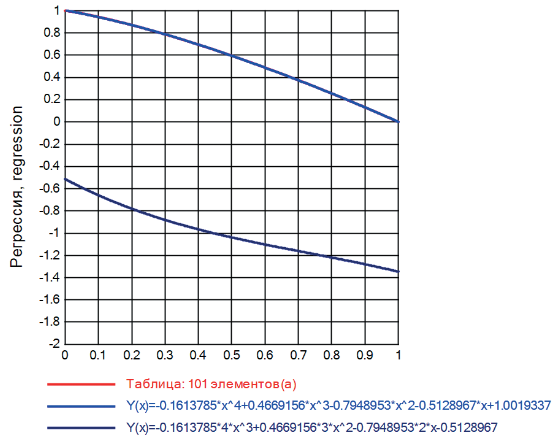


*Figure 2.3. Curve (2.41) (table 2.1, a minimum line $Q_{nm}$), version of approximation by a polynom of the fourth degree and its first derivative on the basis regression the analysis.*

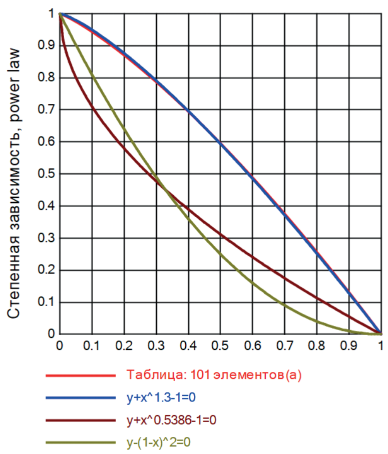


*Figure 2.4. Curve (2.41) (table 2.1) (the minimum line $Q_{nm}$), variants of its approximation by a power law, and also the surface cross section (2.26) with a parabolic cylinder $Q_m - (1 - Q_n)^2 = 0$ (maximum line $Q_{nm}$).*

***Table 2.2.*** ***Coordinates of points of a surface $Q_n$, $Q_m$, $Q_{nm}$ on lines λ = const for a dominating bending of a shell (the expanded version of table 5 [9]).***

| | μ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | λ | | | | | | | | | | | |
| $Q_n$ | 1.0 | 0.00000 | 0.09050 | 0.21897 | 0.34726 | 0.46759 | 0.57813 | 0.67891 | 0.77062 | 0.85414 | 0.93033 | 1.00000 |
| $Q_m$ | | 1.00000 | 0.95024 | 0.85582 | 0.74470 | 0.62830 | 0.51225 | 0.39946 | 0.29140 | 0.18871 | 0.09159 | 0.00000 |
| $Q_{nm}$ | | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| | | | | | | | | | | | | |
| $Q_n$ | 0.9 | 0.00277 | 0.09967 | 0.23533 | 0.36942 | 0.49421 | 0.60812 | 0.71152 | 0.80540 | 0.89130 | 0.98296 | |
| $Q_m$ | | 0.99447 | 0.94107 | 0.84032 | 0.72261 | 0.60010 | 0.47868 | 0.36119 | 0.24887 | 0.14161 | 0.02263 | |
| $Q_{nm}$ | | -0.05249 | -0.04451 | -0.03491 | -0.02663 | -0.01984 | -0.01434 | -0.00989 | -0.00627 | -0.00330 | -0.00049 | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.8 | 0.01235 | 0.11627 | 0.25958 | 0.39974 | 0.52919 | 0.64683 | 0.75362 | 0.85192 | 0.96525 | | |
| $Q_m$ | | 0.97546 | 0.91947 | 0.81312 | 0.68917 | 0.56061 | 0.43347 | 0.31025 | 0.19067 | 0.04585 | | |
| $Q_{nm}$ | | -0.10974 | -0.09179 | -0.07067 | -0.05278 | -0.03831 | -0.02672 | -0.01740 | -0.00979 | -0.00212 | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.7 | 0.03114 | 0.14275 | 0.29415 | 0.44066 | 0.57518 | 0.69753 | 0.81067 | 0.94710 | | | |
| $Q_m$ | | 0.93869 | 0.88187 | 0.77115 | 0.64145 | 0.50666 | 0.37264 | 0.23984 | 0.06914 | | | |
| $Q_{nm}$ | | -0.17098 | -0.14060 | -0.10576 | -0.07681 | -0.05374 | -0.03540 | -0.02056 | -0.00522 | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.6 | 0.06250 | 0.18247 | 0.34238 | 0.49568 | 0.63638 | 0.76701 | 0.92891 | | | | |
| $Q_m$ | | 0.87891 | 0.82409 | 0.71076 | 0.57572 | 0.43364 | 0.28818 | 0.09161 | | | | |
| $Q_{nm}$ | | -0.23438 | -0.18858 | -0.13757 | -0.09607 | -0.06339 | -0.03727 | -0.01015 | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.5 | 0.11111 | 0.24010 | 0.40897 | 0.57016 | 0.72111 | 0.91146 | | | | | |
| $Q_m$ | | 0.79012 | 0.74153 | 0.62771 | 0.48690 | 0.33257 | 0.11168 | | | | | |
| $Q_{nm}$ | | -0.29630 | -0.23153 | -0.16178 | -0.10618 | -0.06235 | -0.01731 | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.4 | 0.18367 | 0.32224 | 0.50097 | 0.67432 | 0.89616 | | | | | | |
| $Q_m$ | | 0.66639 | 0.62989 | 0.51718 | 0.36587 | 0.12669 | | | | | | |
| $Q_{nm}$ | | -0.34985 | -0.26211 | -0.17111 | -0.09927 | -0.02699 | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.3 | 0.28994 | 0.43862 | 0.63097 | 0.88572 | | | | | | | |
| $Q_m$ | | 0.50418 | 0.48669 | 0.37210 | 0.13201 | | | | | | | |
| $Q_{nm}$ | | -0.38234 | -0.26757 | -0.15260 | -0.03900 | | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.2 | 0.44444 | 0.60511 | 0.88564 | | | | | | | | |
| $Q_m$ | | 0.30864 | 0.31363 | 0.11966 | | | | | | | | |
| $Q_{nm}$ | | -0.37037 | -0.22529 | -0.05138 | | | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.1 | 0.66942 | 0.90868 | | | | | | | | | |
| $Q_m$ | | 0.10928 | 0.07641 | | | | | | | | | |
| $Q_{nm}$ | | -0.27047 | -0.05552 | | | | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.0 | 1.00000 | | | | | | | | | | |
| $Q_m$ | | 0.00000 | | | | | | | | | | |
| $Q_{nm}$ | | 0.00000 | | | | | | | | | | |

***Table 2.3*. Coordinates of points of a surface $Q_n$, $Q_m$, $Q_{nm}$ on lines λ = const for a dominating stretching – compression.**

| | μ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | λ | | | | | | | | | | | |
| $Q_n$ | 1.0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| $Q_m$ | | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| $Q_{nm}$ | | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| | | | | | | | | | | | | |
| $Q_n$ | 0.9 | 1.00000 | 0.99999 | 0.99996 | 0.99990 | 0.99980 | 0.99964 | 0.99938 | 0.99889 | 0.99766 | 0.98296 | |
| $Q_m$ | | 0.00000 | 0.00001 | 0.00006 | 0.00014 | 0.00027 | 0.00048 | 0.00082 | 0.00149 | 0.00312 | 0.02263 | |
| $Q_{nm}$ | | 0.00000 | 0.00000 | 0.00000 | 0.00000 | -0.00001 | -0.00001 | -0.00002 | -0.00003 | -0.00007 | -0.00049 | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.8 | 1.00000 | 0.99995 | 0.99978 | 0.99947 | 0.99896 | 0.99810 | 0.99653 | 0.99302 | 0.96525 | | |
| $Q_m$ | | 0.00000 | 0.00007 | 0.00029 | 0.00070 | 0.00138 | 0.00253 | 0.00461 | 0.00927 | 0.04585 | | |
| $Q_{nm}$ | | 0.00000 | 0.00000 | -0.00001 | -0.00003 | -0.00006 | -0.00011 | -0.00021 | -0.00042 | -0.00212 | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.7 | 1.00000 | 0.99985 | 0.99936 | 0.99843 | 0.99683 | 0.99392 | 0.98772 | 0.94710 | | | |
| $Q_m$ | | 0.00000 | 0.00020 | 0.00085 | 0.00207 | 0.00420 | 0.00802 | 0.01619 | 0.06914 | | | |
| $Q_{nm}$ | | 0.00000 | -0.00001 | -0.00006 | -0.00015 | -0.00030 | -0.00058 | -0.00118 | -0.00522 | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.6 | 1.00000 | 0.99964 | 0.99846 | 0.99614 | 0.99182 | 0.98274 | 0.92891 | | | | |
| $Q_m$ | | 0.00000 | 0.00048 | 0.00202 | 0.00506 | 0.01069 | 0.02253 | 0.09161 | | | | |
| $Q_{nm}$ | | 0.00000 | -0.00005 | -0.00021 | -0.00053 | -0.00111 | -0.00237 | -0.01015 | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.5 | 1.00000 | 0.99920 | 0.99654 | 0.99099 | 0.97904 | 0.91146 | | | | | |
| $Q_m$ | | 0.00000 | 0.00103 | 0.00446 | 0.01160 | 0.02693 | 0.11168 | | | | | |
| $Q_{nm}$ | | 0.00000 | -0.00015 | -0.00063 | -0.00166 | -0.00390 | -0.01731 | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.4 | 1.00000 | 0.99826 | 0.99223 | 0.97781 | 0.89616 | | | | | | |
| $Q_m$ | | 0.00000 | 0.00219 | 0.00977 | 0.02780 | 0.12669 | | | | | | |
| $Q_{nm}$ | | 0.00000 | -0.00042 | -0.00188 | -0.00542 | -0.02699 | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.3 | 1.00000 | 0.99604 | 0.98065 | 0.88572 | | | | | | | |
| $Q_m$ | | 0.00000 | 0.00479 | 0.02327 | 0.13201 | | | | | | | |
| $Q_{nm}$ | | 0.00000 | -0.00124 | -0.00612 | -0.03900 | | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.2 | 1.00000 | 0.98930 | 0.88564 | | | | | | | | |
| $Q_m$ | | 0.00000 | 0.01197 | 0.11966 | | | | | | | | |
| $Q_{nm}$ | | 0.00000 | -0.00440 | -0.05138 | | | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.1 | 1.00000 | 0.90868 | | | | | | | | | |
| $Q_m$ | | 0.00000 | 0.07641 | | | | | | | | | |
| $Q_{nm}$ | | 0.00000 | -0.05552 | | | | | | | | | |
| | | | | | | | | | | | | |
| $Q_n$ | 0.0 | 1.00000 | | | | | | | | | | |
| $Q_m$ | | 0.00000 | | | | | | | | | | |
| $Q_{nm}$ | | 0.00000 | | | | | | | | | | |

Table 2.4. Coordinates of points of a surface $Q_n$, $Q_m$, $Q_{nm}$ on lines $\lambda = const$ for a dominating bending of a shell.

| $\lambda$ | $\mu_{min}$ | $\mu$ | $Q_n$ | $Q_m$ | $Q_{nm}$ |
|---|---|---|---|---|---|
| | | | | | |
| 1.0 | 0.0 | 0.00000 | 0.00000 | 1.00000 | 0.00000 |
| 1.0 | 0.1 | 0.10000 | 0.09050 | 0.95024 | 0.00000 |
| 1.0 | 0.2 | 0.20000 | 0.21897 | 0.85582 | 0.00000 |
| 1.0 | 0.3 | 0.30000 | 0.34726 | 0.74470 | 0.00000 |
| 1.0 | 0.4 | 0.40000 | 0.46759 | 0.62830 | 0.00000 |
| 1.0 | 0.5 | 0.50000 | 0.57813 | 0.51225 | 0.00000 |
| 1.0 | 0.6 | 0.60000 | 0.67891 | 0.39946 | 0.00000 |
| 1.0 | 0.7 | 0.70000 | 0.77062 | 0.29140 | 0.00000 |
| 1.0 | 0.8 | 0.80000 | 0.85414 | 0.18871 | 0.00000 |
| 1.0 | 0.9 | 0.90000 | 0.93033 | 0.09159 | 0.00000 |
| 1.0 | 1.0 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| | | | | | |
| 0.9 | 0.0 | 0.05000 | 0.00277 | 0.99447 | -0.05249 |
| 0.9 | 0.1 | 0.11190 | 0.09967 | 0.94107 | -0.04451 |
| 0.9 | 0.2 | 0.20640 | 0.23533 | 0.84032 | -0.03491 |
| 0.9 | 0.3 | 0.30460 | 0.36942 | 0.72261 | -0.02663 |
| 0.9 | 0.4 | 0.40380 | 0.49421 | 0.60010 | -0.01984 |
| 0.9 | 0.5 | 0.50350 | 0.60812 | 0.47868 | -0.01434 |
| 0.9 | 0.6 | 0.60350 | 0.71152 | 0.36119 | -0.00989 |
| 0.9 | 0.7 | 0.70390 | 0.80540 | 0.24887 | -0.00627 |
| 0.9 | 0.8 | 0.80550 | 0.89130 | 0.14161 | -0.00330 |
| 0.9 | 0.9 | 0.92600 | 0.98296 | 0.02263 | -0.00049 |
| | | | | | |
| 0.8 | 0.0 | 0.10000 | 0.01235 | 0.97546 | -0.10974 |
| 0.8 | 0.1 | 0.14190 | 0.11627 | 0.91947 | -0.09179 |
| 0.8 | 0.2 | 0.22480 | 0.25958 | 0.81312 | -0.07067 |
| 0.8 | 0.3 | 0.31820 | 0.39974 | 0.68917 | -0.05278 |
| 0.8 | 0.4 | 0.41530 | 0.52919 | 0.56061 | -0.03831 |
| 0.8 | 0.5 | 0.51440 | 0.64683 | 0.43347 | -0.02672 |
| 0.8 | 0.6 | 0.61510 | 0.75362 | 0.31025 | -0.01740 |
| 0.8 | 0.7 | 0.71880 | 0.85192 | 0.19067 | -0.00979 |
| 0.8 | 0.8 | 0.85440 | 0.96525 | 0.04585 | -0.00212 |
| | | | | | |
| 0.7 | 0.0 | 0.15000 | 0.03114 | 0.93869 | -0.17098 |
| 0.7 | 0.1 | 0.18120 | 0.14275 | 0.88187 | -0.14060 |
| 0.7 | 0.2 | 0.25270 | 0.29415 | 0.77115 | -0.10576 |
| 0.7 | 0.3 | 0.34030 | 0.44066 | 0.64145 | -0.07681 |
| 0.7 | 0.4 | 0.43500 | 0.57518 | 0.50666 | -0.05374 |

Table 2.5. Coordinates of points of a surface $Q_n$, $Q_m$, $Q_{nm}$ on lines $\lambda = const$ for a dominating stretching – compression.

| $\lambda$ | $\mu_{min}$ | $\mu$ | $Q_n$ | $Q_m$ | $Q_{nm}$ |
|---|---|---|---|---|---|
| | | | | | |
| 1.0 | 0.0 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.1 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.2 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.3 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.4 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.5 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.6 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.7 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.8 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 0.9 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| 1.0 | 1.0 | 1.00000 | 1.00000 | 0.00000 | 0.00000 |
| | | | | | |
| 0.9 | 0.0 | 0.95000 | 1.00000 | 0.00000 | 0.00000 |
| 0.9 | 0.1 | 0.95000 | 0.99999 | 0.00001 | 0.00000 |
| 0.9 | 0.2 | 0.94990 | 0.99996 | 0.00006 | 0.00000 |
| 0.9 | 0.3 | 0.94990 | 0.99990 | 0.00014 | 0.00000 |
| 0.9 | 0.4 | 0.94970 | 0.99980 | 0.00027 | -0.00001 |
| 0.9 | 0.5 | 0.94950 | 0.99964 | 0.00048 | -0.00001 |
| 0.9 | 0.6 | 0.94910 | 0.99938 | 0.00082 | -0.00002 |
| 0.9 | 0.7 | 0.94840 | 0.99889 | 0.00149 | -0.00003 |
| 0.9 | 0.8 | 0.94670 | 0.99766 | 0.00312 | -0.00007 |
| 0.9 | 0.9 | 0.92600 | 0.98296 | 0.02263 | -0.00049 |
| | | | | | |
| 0.8 | 0.0 | 0.90000 | 1.00000 | 0.00000 | 0.00000 |
| 0.8 | 0.1 | 0.89990 | 0.99995 | 0.00007 | 0.00000 |
| 0.8 | 0.2 | 0.89970 | 0.99978 | 0.00029 | -0.00001 |
| 0.8 | 0.3 | 0.89930 | 0.99947 | 0.00070 | -0.00003 |
| 0.8 | 0.4 | 0.89860 | 0.99896 | 0.00138 | -0.00006 |
| 0.8 | 0.5 | 0.89740 | 0.99810 | 0.00253 | -0.00011 |
| 0.8 | 0.6 | 0.89540 | 0.99653 | 0.00461 | -0.00021 |
| 0.8 | 0.7 | 0.89070 | 0.99302 | 0.00927 | -0.00042 |
| 0.8 | 0.8 | 0.85440 | 0.96525 | 0.04585 | -0.00212 |
| | | | | | |
| 0.7 | 0.0 | 0.85000 | 1.00000 | 0.00000 | 0.00000 |
| 0.7 | 0.1 | 0.84980 | 0.99985 | 0.00020 | -0.00001 |
| 0.7 | 0.2 | 0.84920 | 0.99936 | 0.00085 | -0.00006 |
| 0.7 | 0.3 | 0.84800 | 0.99843 | 0.00207 | -0.00015 |
| 0.7 | 0.4 | 0.84600 | 0.99683 | 0.00420 | -0.00030 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.7 | 0.5 | 0.53420 | 0.69753 | 0.37264 | -0.03540 | | 0.7 | 0.5 | 0.84240 | 0.99392 | 0.00802 | -0.00058 |
| 0.7 | 0.6 | 0.63900 | 0.81067 | 0.23984 | -0.02056 | | 0.7 | 0.6 | 0.83470 | 0.98772 | 0.01619 | -0.00118 |
| 0.7 | 0.7 | 0.78580 | 0.94710 | 0.06914 | -0.00522 | | 0.7 | 0.7 | 0.78580 | 0.94710 | 0.06914 | -0.00522 |
| | | | | | | | | | | | | |
| 0.6 | 0.0 | 0.20000 | 0.06250 | 0.87891 | -0.23438 | | 0.6 | 0.0 | 0.80000 | 1.00000 | 0.00000 | 0.00000 |
| 0.6 | 0.1 | 0.22510 | 0.18247 | 0.82409 | -0.18858 | | 0.6 | 0.1 | 0.79960 | 0.99964 | 0.00048 | -0.00005 |
| 0.6 | 0.2 | 0.28790 | 0.34238 | 0.71076 | -0.13757 | | 0.6 | 0.2 | 0.79820 | 0.99846 | 0.00202 | -0.00021 |
| 0.6 | 0.3 | 0.37040 | 0.49568 | 0.57572 | -0.09607 | | 0.6 | 0.3 | 0.79550 | 0.99614 | 0.00506 | -0.00053 |
| 0.6 | 0.4 | 0.46370 | 0.63638 | 0.43364 | -0.06339 | | 0.6 | 0.4 | 0.79050 | 0.99182 | 0.01069 | -0.00111 |
| 0.6 | 0.5 | 0.56690 | 0.76701 | 0.28818 | -0.03727 | | 0.6 | 0.5 | 0.78010 | 0.98274 | 0.02253 | -0.00237 |
| 0.6 | 0.6 | 0.72110 | 0.92891 | 0.09161 | -0.01015 | | 0.6 | 0.6 | 0.72110 | 0.92891 | 0.09161 | -0.01015 |
| | | | | | | | | | | | | |
| 0.5 | 0.0 | 0.25000 | 0.11111 | 0.79012 | -0.29630 | | 0.5 | 0.0 | 0.75000 | 1.00000 | 0.00000 | 0.00000 |
| 0.5 | 0.1 | 0.27160 | 0.24010 | 0.74153 | -0.23153 | | 0.5 | 0.1 | 0.74910 | 0.99920 | 0.00103 | -0.00015 |
| 0.5 | 0.2 | 0.32860 | 0.40897 | 0.62771 | -0.16178 | | 0.5 | 0.2 | 0.74630 | 0.99654 | 0.00446 | -0.00063 |
| 0.5 | 0.3 | 0.40830 | 0.57016 | 0.48690 | -0.10618 | | 0.5 | 0.3 | 0.74050 | 0.99099 | 0.01160 | -0.00166 |
| 0.5 | 0.4 | 0.50500 | 0.72111 | 0.33257 | -0.06235 | | 0.5 | 0.4 | 0.72800 | 0.97904 | 0.02693 | -0.00390 |
| 0.5 | 0.5 | 0.66140 | 0.91146 | 0.11168 | -0.01731 | | 0.5 | 0.5 | 0.66140 | 0.91146 | 0.11168 | -0.01731 |
| | | | | | | | | | | | | |
| 0.4 | 0.0 | 0.30000 | 0.18367 | 0.66639 | -0.34985 | | 0.4 | 0.0 | 0.70000 | 1.00000 | 0.00000 | 0.00000 |
| 0.4 | 0.1 | 0.31990 | 0.32224 | 0.62989 | -0.26211 | | 0.4 | 0.1 | 0.69830 | 0.99826 | 0.00219 | -0.00042 |
| 0.4 | 0.2 | 0.37460 | 0.50097 | 0.51718 | -0.17111 | | 0.4 | 0.2 | 0.69260 | 0.99223 | 0.00977 | -0.00188 |
| 0.4 | 0.3 | 0.45700 | 0.67432 | 0.36587 | -0.09927 | | 0.4 | 0.3 | 0.67910 | 0.97781 | 0.02780 | -0.00542 |
| 0.4 | 0.4 | 0.60830 | 0.89616 | 0.12669 | -0.02699 | | 0.4 | 0.4 | 0.60830 | 0.89616 | 0.12669 | -0.02699 |
| | | | | | | | | | | | | |
| 0.5 | 0.5 | 0.66140 | 0.91146 | 0.11168 | -0.01731 | | 0.5 | 0.5 | 0.66140 | 0.91146 | 0.11168 | -0.01731 |
| | | | | | | | | | | | | |
| 0.4 | 0.0 | 0.30000 | 0.18367 | 0.66639 | -0.34985 | | 0.4 | 0.0 | 0.70000 | 1.00000 | 0.00000 | 0.00000 |
| 0.4 | 0.1 | 0.31990 | 0.32224 | 0.62989 | -0.26211 | | 0.4 | 0.1 | 0.69830 | 0.99826 | 0.00219 | -0.00042 |
| 0.4 | 0.2 | 0.37460 | 0.50097 | 0.51718 | -0.17111 | | 0.4 | 0.2 | 0.69260 | 0.99223 | 0.00977 | -0.00188 |
| 0.4 | 0.3 | 0.45700 | 0.67432 | 0.36587 | -0.09927 | | 0.4 | 0.3 | 0.67910 | 0.97781 | 0.02780 | -0.00542 |
| 0.4 | 0.4 | 0.60830 | 0.89616 | 0.12669 | -0.02699 | | 0.4 | 0.4 | 0.60830 | 0.89616 | 0.12669 | -0.02699 |
| | | | | | | | | | | | | |
| 0.3 | 0.0 | 0.35000 | 0.28994 | 0.50418 | -0.38234 | | 0.3 | 0.0 | 0.65000 | 1.00000 | 0.00000 | 0.00000 |
| 0.3 | 0.1 | 0.36980 | 0.43862 | 0.48669 | -0.26757 | | 0.3 | 0.1 | 0.64670 | 0.99604 | 0.00479 | -0.00124 |
| 0.3 | 0.2 | 0.42770 | 0.63097 | 0.37210 | -0.15260 | | 0.3 | 0.2 | 0.63410 | 0.98065 | 0.02327 | -0.00612 |
| 0.3 | 0.3 | 0.56350 | 0.88572 | 0.13201 | -0.03900 | | 0.3 | 0.3 | 0.56350 | 0.88572 | 0.13201 | -0.03900 |
| | | | | | | | | | | | | |
| 0.2 | 0.0 | 0.40000 | 0.44444 | 0.30864 | -0.37037 | | 0.2 | 0.0 | 0.60000 | 1.00000 | 0.00000 | 0.00000 |
| 0.2 | 0.1 | 0.42290 | 0.60511 | 0.31363 | -0.22529 | | 0.2 | 0.1 | 0.59260 | 0.98930 | 0.01197 | -0.00440 |
| 0.2 | 0.2 | 0.52920 | 0.88564 | 0.11966 | -0.05138 | | 0.2 | 0.2 | 0.52920 | 0.88564 | 0.11966 | -0.05138 |
| | | | | | | | | | | | | |
| 0.1 | 0.0 | 0.45000 | 0.66942 | 0.10928 | -0.27047 | | 0.1 | 0.0 | 0.55000 | 1.00000 | 0.00000 | 0.00000 |
| 0.1 | 0.1 | 0.50740 | 0.90868 | 0.07641 | -0.05552 | | 0.1 | 0.1 | 0.50740 | 0.90868 | 0.07641 | -0.05552 |
| | | | | | | | | | | | | |
| 0.0 | 0.0 | 0.50000 | 1.00000 | 0.00000 | 0.00000 | | 0.0 | 0.0 | 0.50000 | 1.00000 | 0.00000 | 0.00000 |

In fig. 2.4 shows the curve (2.41) (table 2.1) (the minimum line $Q_{nm}$), variants of its approximation by a power law, and also the surface cross section (2.26) with a parabolic cylinder $Q_m - (1 - Q_n)^2 = 0$ (maximum line $Q_{nm}$). Other variants of approximation are given in part 3 of the article.

## CONCLUSIONS

Alternative dependences of the finite relationship are developed, their equivalence to the relations A.A. Ilyushin is proved, approximate dependences of the final relationship are obtained. Based on the regression analysis of the minimum line , variants of its approximation by algebraic polynomials are obtained.

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