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FORMULAS FOR CALCULATING THE DEFLECTIONS OF A TRIANGULAR TRUSS OF SPATIAL COVER*

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Statement of the problem. The scheme of triangular in terms of dome type cover is proposed. The construction is statically determinate. The formulas for the dependence of the deflection on the number of panels and sizes are derived by generalizing a series of individual solutions by induction.
Materials and methods. The forces in the coating rods are performed by cutting nodes in symbolic form using operators of the Maple symbolic mathematics system. The unknown systems of equilibrium equations in projections on the coordinate axis include the reactions of vertical supports located on the sides of the truss. One of the corners of the truss also has a spherical support, one is cylindrical. The Maxwell—Mohr formula is used to calculate the deflection of the vertex. The analysis of sequences of coefficients in solutions for individual trusses with different numbers of panels yields expressions for common terms included in the desired calculation formula.
Results. Formulas for the dependence of the deflections of the truss on the number of panels for a vertical load evenly distributed over the nodes of the truss and a horizontal wind load applied to one of the sides of the structure are obtained. The solutions have a simple polynomial form. The curves of the dependence of the horizontal displacement of the dome top on the number of panels reveal a minimum. Asymptotics of the solutions is identified.
Conclusions. A scheme of a statically determinate symmetric spatial dome is developed and its mathematical model is constructed, allowing analytical solutions with an arbitrary number of panels. The identified dependencies can be used both to evaluate the accuracy of numerical solutions and to find optimal combinations of structural dimensions in terms of rigidity.

Keywords: triangular dome, spatial truss, deflection, induction, *Maple*, analytical solution.

Introduction. Spatial bar structures normally contain a large number of elements, which causes some difficulties in numerical calculation. If the most common finite element method is employed in addressing such problems [10, 19, 23, 24], the inevitable error associated with the accumulation of rounding errors compromises accuracy [3]. This feature manifests itself with a very large number of elements, e.g., in trusses of large-span bridges and roofs of buildings and structures. If the structure is regular and a periodically repeating group of rods can be distinguished in it, analytical methods can be utilized to overcome the “curse of dimensionality” and evaluate its stress-strain in the calculation.

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Analytical methods are known and widely used in practice where numerical calculations based on variational principles in the system of symbolic mathematics are substituted by analytical ones with the preservation of the calculation algorithm [11, 12, 20]. This method is applicable to both regular and arbitrary systems. The inductive analytical method for calculating structures is applicable to regular systems with periodically repeating structures and normally leads to simple calculation formulas [9]. This method is based on the generalization of a series of solutions for a construction with different orders and is used mainly for statically determined systems. The problem of existence and analysis of statically defined regular trusses was first raised by R. G. Hutchinson, N. A. Fleck, F. W. Zok [13, 14, 26]. The reference books [17, 18] provide diagrams of flat statically determinate regular trusses and formulas for calculating their deflection under the action of several types of loads. There are also formulas for calculating arched [8, 15, 22] and lattice trussed and cantilever trusses [1, 4, 5, 7, 21, 25]. The method of induction is also applicable for calculating the frequencies of natural oscillations of regular structures [2]. A review of analytical calculations of regular trusses based on the method of induction with the involvement of a computer mathematics system is found in [6].

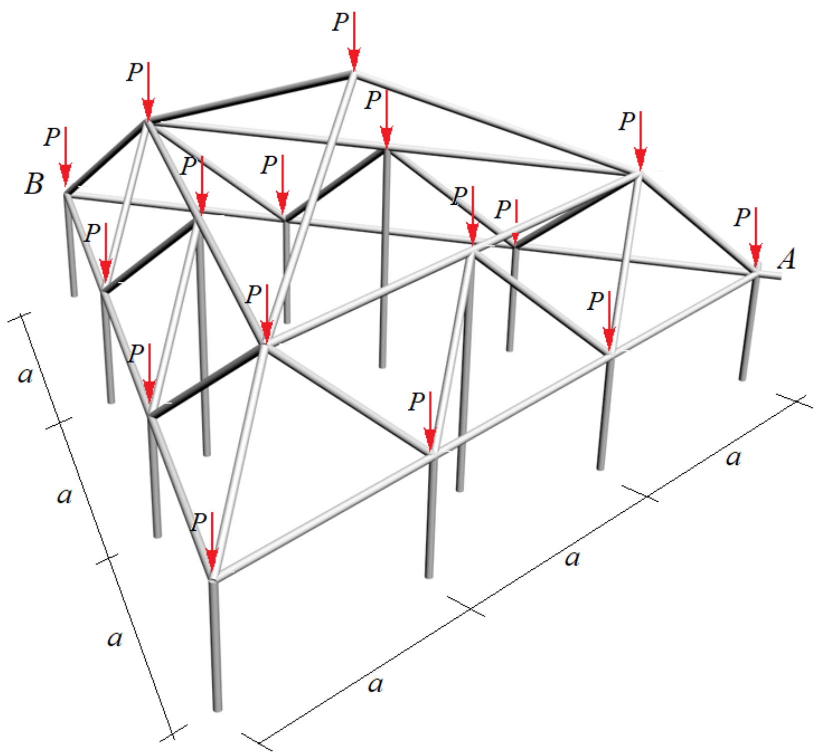

Fig. 1. Truss under the action of a vertical distributed nodal load, $n = 3$

In this paper, we consider a domed truss (Fig. 1, 2) consisting of horizontal triangular contours connected by braces and supported by vertical posts with a height h on the lower contour and $2h$ on the upper one. The trihedral dome of height H rests on the upper contour. The length of the braces connecting the contours is $\sqrt{a^2/3+h^2}$. The trihedral dome of height H rests on the upper contour. The length of the braces connecting the contours is. The connections of the rods in the farm are articulated. In corner A there is a spherical support hinge modeled by three mutually perpendicular rods, in corner B there is a cylindrical support corresponding to two rods. The truss contains $n_s = 18n - 6$ rods, including $6n - 3$ vertical support posts. An inductive method for deriving calculation formulas for an arbitrary number of panels is applicable to the farm. Deformations and the first frequency of a similar spatial structure are calculated in an analytical form in [16].

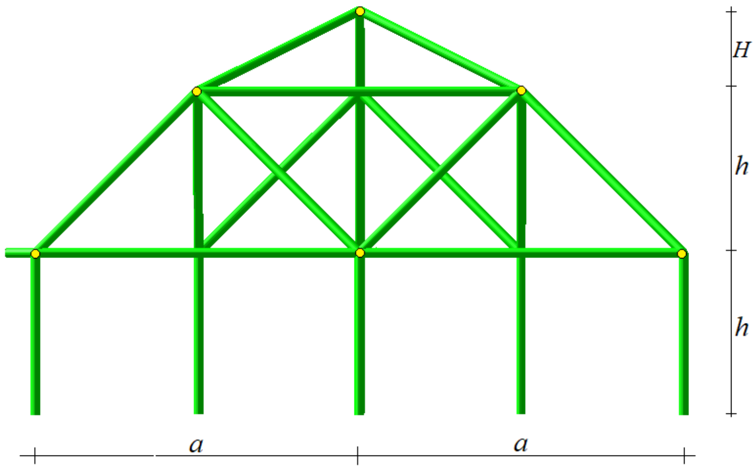

Fig. 2. Gird sizes, n=2, $H = (n-1)h/2$

1. Calculation of the efforts. We will calculate the major characteristics of the stress-strain using the *Maple* symbolic mathematics system [9]. The modeling of the truss scheme begins with the input of the coordinates of the nodes and the structure of the bar lattice.

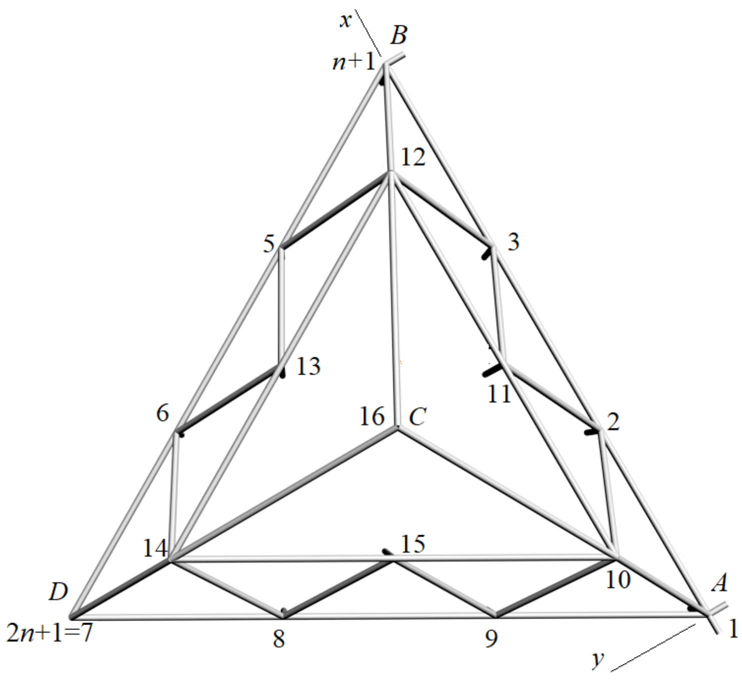

Fig. 3. Farm node numbering. Vertex numbers A, B, C and D: $n+1$, 1, $6n-2$, $n+1$

The coordinates of the nodes of the lower contour take the following form:

$$x_i = a(i-1), y_i = 0,\ \ z_i = 0,$$
$$x_{i+n} = a(2n-i+1)/2, y_{i+n} = a(i-1)\sqrt{3}/2,\ z_{i+n} = 0,$$
$$x_{i+2n} = a(n-i+1)/2, y_{i+2n} = a(n-i+1)\sqrt{3}/2,\ z_{i+2n} = 0,\ i = 1,..,n.$$

The coordinates of the nodes of the upper (smaller) contour are as follows:

$$x_{i+3n} = a(2i-1)/2, y_{i+3n} = a(i-1)\sqrt{3}/6,\ z_{i+n} = h,$$
$$x_{i+4n-1} = a(2n-i)/2, y_{i+4n-1} = a(3i-2)\sqrt{3}/6,\ z_{i+4n-1} = h,$$
$$x_{i+5n-2} = a(n-i+1)/2, y_{i+5n-2} = a(3n-3i+1)\sqrt{3}/6,\ z_{i+5n-2} = h,\ i = 1,..,n-1.$$

The lattice configuration is introduced using special ordered lists of the numbers of the ends of the corresponding rods, by analogy with the assignment of graphs in discrete mathematics. According to the data on the coordinates of the nodes and the structure of the connection of the rods, the direction cosines of the forces are found in the equilibrium equations of the nodes in projections on the coordinate axes. We write the system of equilibrium equations in the form $\mathbf{GS} = \mathbf{B}$ where $\mathbf{G}$ is a matrix of coefficients (direction cosines of the forces calculated from the coordinates of the nodes), $\mathbf{S}$ is a vector of forces and reactions of supports, $\mathbf{B}$ is a loading vector. In the elements such as B_{3i-2} where i is the number of the node, the loads on the node i are recorded in the projection on the x axis. The elements B_{3i-1} contain projections of forces on the y-axis. Vertical loads are written in the elements B_{3i}.

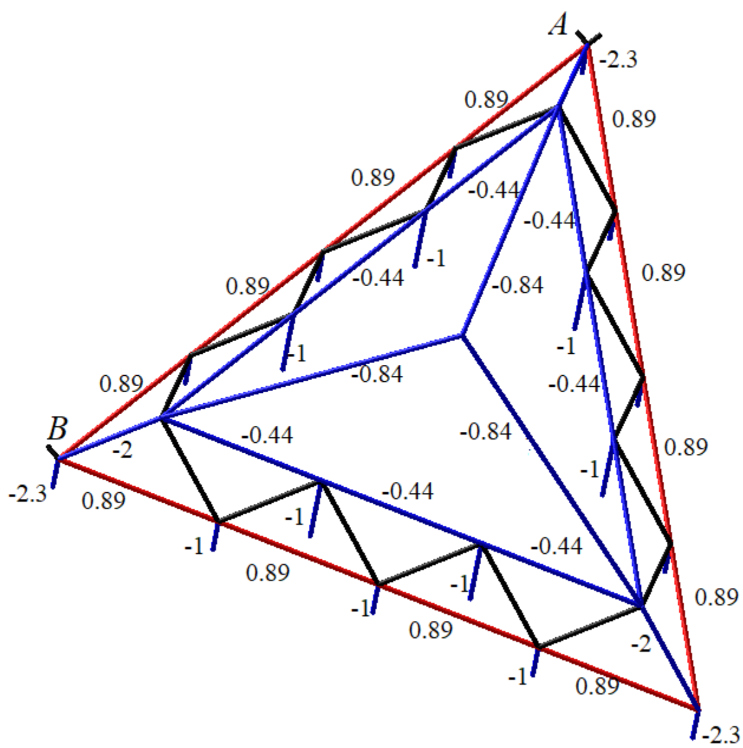

Fig. 4. Forces in rods
from the action of a distributed load

In case of a uniform vertical (Fig. 1) nodal load, non-zero elements of the load vector take the following form: $B_{3i} = P,\ i = 1,...,6n-2.$ The numerical calculation of forces provide a pattern of the distribution of forces shown in Fig. 4.

In the calculations it is accepted that $n = 4,\ a = 10$ m, $h = 5$ m. The forces in the rods are related to the load P on the node, the compressed rods are marked in blue, the tension rods are marked in red. The lower belt turned out to be stretched, the upper — compressed. The braces are marked in black, with such a load they are not stressed. The reactions of three corner supports from the action of a distributed load do not depend on the number of panels: $R_o = 7P/3$, the reactions of the supports, except for the angular ones along the lower contour and the supports along the upper contour, are the same and also do not depend on the order of the system: $R_1 = P$. If the force P is applied to the top of the truss (point C), then the corresponding reactions from its action are $R_O = P/3$ and $R_1 = 0$. This regular structure is characterized by the fact that the forces in it for given loads do not depend on the number of panels. If we consider the actions of a horizontal (wind) load, which is relevant for a dome-type structure with a large sail area (Fig. 5), the pattern of force distribution will already depend on the number of panels (Fig. 6). Let us cnsider a load directed along the y axis. The load is conditionally distributed evenly over the nodes of the windward side. The load vector in the system of equilibrium equations takes the following form:

$$B_{3i-1} = P,\ i = 6n-2,\ i = 1,...,n+1,\ i = 3n+1,...,4n.$$

In case when flat shields are mounted on the edges of the structure, and the load is uniform over the surface of the truss, the magnitude of the forces applied to the nodes will depend on the areas of the faces.

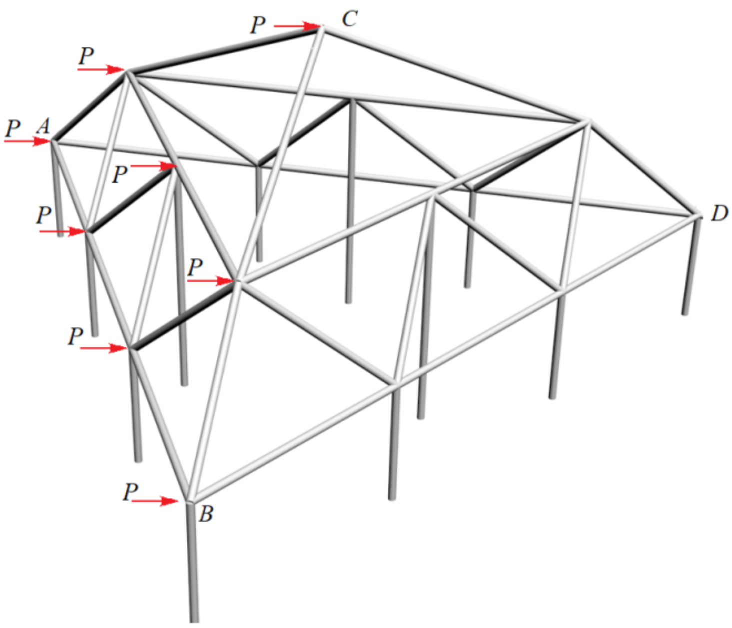

Fig. 5. Lateral (wind) load to the gird

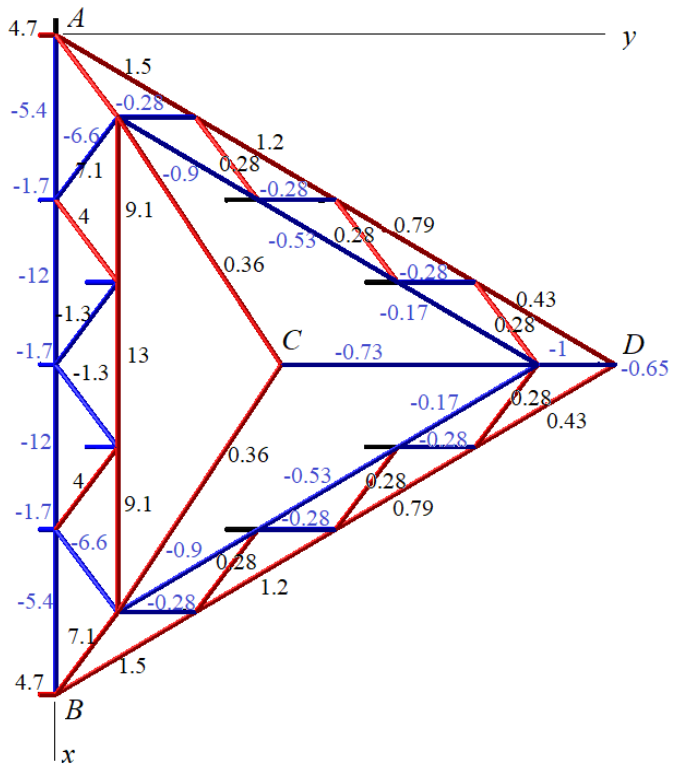

Fig. 6. Distribution of efforts from the action of the wind load along the y axis at $n = 4$, $a = 10$ м, $h = 5$ м

Calculating the forces and reactions of the structure supports for different n, we obtain sequences whose common members can be identified using *Maple*. For the vertical reaction of the support on the windward side (holding, directed downwards), we have the following sequence:

$$R_A(2) = 19P\sqrt{3}h/(12a),\ R_A(3) = 31\sqrt{3}Ph/(9a),\ R_A(4) = 43\sqrt{3}hP/(8a),$$
$$R_A(5) = 22\sqrt{3}hP/(3a),\ R_A(6) = 335\sqrt{3}hP/(36a),...$$

The general term of this sequence takes the form $R_A(n) = \sqrt{3}\,hP(12n-5)(n-1)/(6na)$. It is clear from the symmetry $R_A = R_B$. Similarly, the dependence on the number of panels and the reaction of the support D on the leeward side are found $R_D(n) = \sqrt{3}\,hP(n+5)/(3na)$. The reaction is directed upwards, the support rod is compressed. The reactions of the vertical supports on the windward side, as well as in the case of a vertical load, do not depend on the number of panels, are directed upwards and are equal to $2\sqrt{3}hP/a$. The reactions of all other vertical supports (on the sides AD and BD) are equal to zero for any number of panels.

2. Deflection. Vertical load. The constructed mathematical model of the structure with the calculation of the forces in the rods makes it easy to determine the displacements of the nodes, e.g., the deflection of the top C and its displacement under the action of various loads. The deflection is searched for by the Maxwell-Mohr formula is written as follows:

$$\Delta = P\sum_{j=1}^{n_S} \frac{S_j s_j l_j}{EF}, \tag{1}$$

where E is an elastic modulus of the rods; F is the area of the section; l_j and S_j are the length and force in the j-th rod from the action of the load; s_j is the force from a single vertical force applied to the top C, if its deflection (vertical displacement) is identified. Sequential calculation of the deflection of a truss with a different number of panels from the action of a uniform vertical nodal load yields the following results:

$$\Delta = P(36a^3 + 8c^3 + 378h^3 + d^3)/(162h^2EF),$$
$$\Delta = P(24a^3 + 4c^3 + 189h^3 + d^3)/(81h^2EF),$$
$$\Delta = P(60a^3 + 8c^3 + 378h^3 + 3d^3)/(162h^2EF),$$
$$\Delta = P(36a^3 + 4c^3 + 189h^3 + 2d^3)/(81h^2EF),\ \ldots .$$

In the general case we have the form of a formula for deflection:

$$\Delta = P\left(C_1a^3 + C_2c^3 + C_3h^3 + C_4d^3\right)/\left(h^2EF\right), \tag{2}$$

where $c = \sqrt{3a^2 + 9h^2}$, $d = \sqrt{12a^2 + 9h^2}$. The coefficients at the degrees of truss sizes form sequences whose common members can be found using *Maple*. While processing the solution $n = 2,3,...,7$, we get the following result:

$$C_1 = 2(n+1)/27,\ C_2 = 4/81,\ C_3 = 7/3,\ C_4 = (n-1)/162. \tag{3}$$

Compared to similar solutions that have dependences of the coefficients on the number of panels in the form of polynomials up to the fifth order [17, 18], the resulting linear solution is much simpler. This is due to the discovered rare feature of the stress state of the structure under the action of a uniform load: the forces in the rods and the reactions of the supports do not depend on the number of panels. The same simple solution is obtained in the case when the truss is loaded with only one vertical force applied to the vertex C. The coefficients in (2) for such a load are as follows:

$$C_1 = (2n-1)/27,\ C_2 = 1/81,\ C_3 = 1/3,\ C_4 = (n-1)/162. \tag{4}$$

The linear combination of formula (2) with coefficients (3) and (4) expands the class of loads used in the solution.

3. Deflection. Wind load. Under the deflection in case of a horizontal load, we mean the horizontal displacement of the vertex C. To determine this value, we use the Maxwell-Mohr formula (1) where s_j is the effort from a unit horizontal force applied to the vertex C and S_j is the effort in the j-th rod from the action of a horizontal load. The type of solution differs by (2) denominator

$$\Delta = P\left(C_1a^3 + C_2c^3 + C_3h^3 + C_4d^3\right)/\left(a^2EF\right). \tag{5}$$

Using the induction method, based on the results of calculating eight trusses, we obtain the coefficients:

$$
\begin{aligned}
C_1 &= (6n^4 + 2n^3 + 25n^2 + 4n + 5)/(36n),\\
C_2 &= (12n^3 - 3n^2 + 10n + 5)/(162n^2),\\
C_3 &= (n+1)(4n^2 - 5n + 5)/(2n^2),\\
C_4 &= (n-1)/108.
\end{aligned}
\tag{6}
$$

The solution in this case turns out to be much more complicated than for the vertical load, not only because the expressions depend nonlinearly on n, but also because both numerators and denominators depend on n in the coefficients.

4. Analysis of the solutions. Let us plot the solution graphs (2), (3) for the vertical load. Let us fix the length of the side of the cover $L = na$ and the total loading $P_{sum} = P(6n-2)$. Let us introduce the value of the dimensionless deflection $\Delta' = \Delta EF/(P_{sum}L)$. The dependence of the deflection on the number of panels (Fig. 7) is monotonic. There is a horizontal asymptote $\lim_{n\to\infty} \Delta' = h/(36L)$.

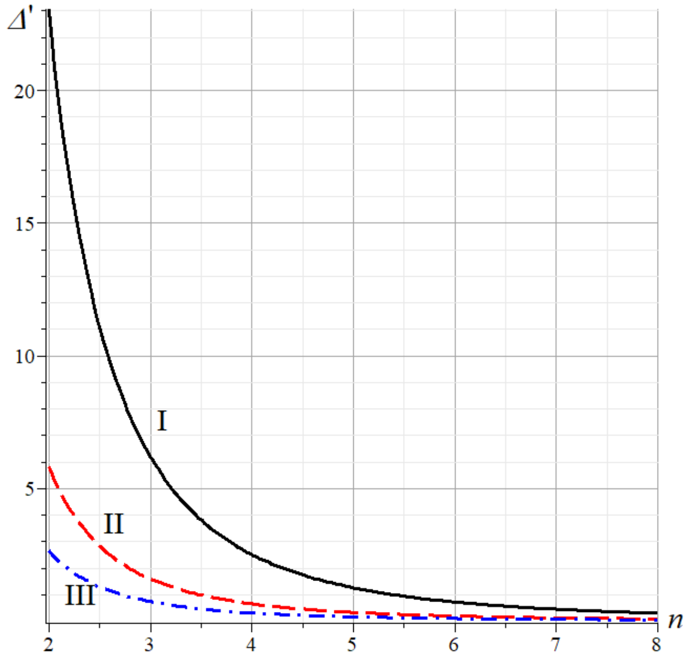

Fig. 7. Dependence of the deflection on the number of panels under vertical load, L = 50 m:
I — h = 1 m; II — h = 2 m; III — h = 3 m

The graphs of the solution of the problem of the deflection of the top for the wind load (5), (6), constructed under the same assumptions as the curves in graph 7, show a minimum (Fig. 8). As the height of the truss increases, the minimum shifts towards a smaller number of panels. *Maple* tools can also be used to find the nonlinear (quadratic) asymptotics of the solution $\lim_{n\to\infty} \Delta'/n^2 = 17h^3/(24L^3)$.

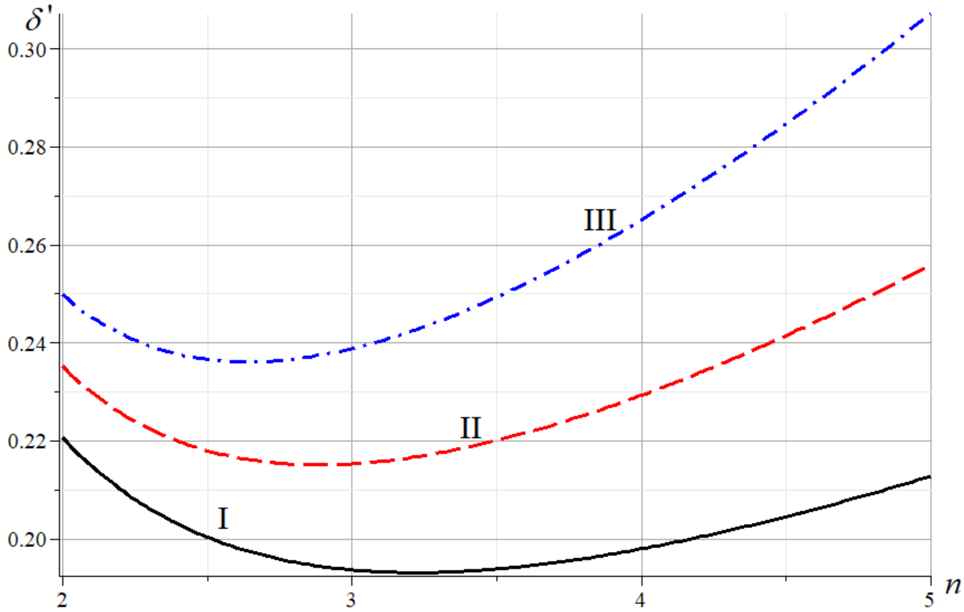

Fig. 8. Dependence of the horizontal displacement of the top C on the number of panels under wind load, $L = 25$ m: I — $h = 1$ m; II — $h = 3$ m; III — $h = 4$ m

Conclusions. A scheme of a spatial statically determinate regular dome structure is set forth, which allows an analytical solution of the deflection problem for an arbitrary number of panels. The formulas for reactions of supports and deflections for three types of loads are obtained. It is typical that for a uniform vertical nodal load, the distribution and values of the forces in the rods do not depend on the number of panels. The resulting horizontal asymptote for the deflection in this case shows that an increase in the number of panels at a fixed length of the side of the structure and the total load does not lead to a significant decrease in the deflection.

The structures of the proposed type can be used as coverings for public and industrial buildings, and the analytical solutions make it possible to evaluate the accuracy of numerical solutions in the case of a more complex structure.

References

1. Belyankin N. A., Boiko A. Yu. Formuly dlya progiba balochnoi fermy s proizvol'nym chislom panelei pri ravnomernom zagruzhenii [Formulas for deflection of a girder truss with an arbitrary number of panels at equal loading]. *Stroitel'naya mekhanika i konstruktsii*, 2019, no. 1 (20), pp. 21—29.
2. Vorob'ev O. V. O metodakh polucheniya analiticheskogo resheniya dlya problemy sobstvennykh chastot sharnirnykh konstruktsii [On methods of obtaining an analytical solution for the problem of natural frequencies of articulated structures]. *Stroitel'naya mekhanika i konstruktsii*, 2020, no. 1 (24), pp. 25—38.
3. Rogalev A. N., Doronin S. V. Ispol'zovanie kriteriev obuslovlennosti pri chislennykh raschetakh napryazhennogo sostoyaniya silovykh konstruktsii [The use of conditionality criteria in numerical calculations of the stress state of power structures]. *Sistemy. Metody. Tekhnologii*, 2016, no. 2, pp. 91—99. DOI: 10.18324/2077-5415-2016-2-91-99.
4. Sud I. B. Formuly dlya progiba shprengel'noi balochnoi fermy s proizvol'nym chislom panelei [Formulas for deflection of a sprengel beam truss with an arbitrary number of panels]. *Stroitel'naya mekhanika i konstruktsii*, 2020, no. 2 (25), pp. 25—32.
5. Terze S. V. Analiticheskii raschet zavisimosti deformatsii konsol'noi stoiki ot chisla panelei v sisteme Maple [Analytical calculation of the dependence of deformations of a cantilever rack on the number of panels in the Maple system]. *Stroitel'naya mekhanika i konstruktsii*, 2020, no. 2 (25), pp. 16—24.

6. Tin'kov D. V. Sravnitel'nyi analiz analiticheskikh reshenii zadachi o progibe fermennykh konstruktsii [Comparative analysis of analytical solutions to the problem of deflection of truss structures]. *Inzhenerno-stroitel'nyi zhurnal*, 2015, no. 5 (57), pp. 66—73. DOI: 10.5862/MCE.57.6.
7. Tkachuk G. N. Formula zavisimosti progiba nesimmetrichno nagruzhennoi ploskoi fermy s usi-lennymi raskosami ot chisla panelei [The formula for the dependence of the deflection of an asymmetrically loaded flat truss with increased braces on the number of panels]. *Stroitel'naya mekhanika i konstruktsii*, 2019, no. 2 (21), pp. 32—39.
8. Bolotina T. D. The deflection of the flat arch truss with a triangular lattice depending on the number of panels. *Bulletin of Scientific Conferences*, 2016, no. 4—3 (8), pp. 7—8.
9. Buka-Vaivade K., Kirsanov M. N., Serdjuks D. O. Calculation of deformations of a cantilever-frame planar truss model with an arbitrary number of panels. Vestnik MGSU, 2020, no. 4, pp. 510—517. DOI: 10.22227/1997-0935.2020.4.510-517.
10. Dong L. Mechanical responses of snap-fit Ti-6Al-4V warren-truss lattice structures. *International Journal of Mechanical Sciences*, 2020, vol. 173, p. 105460. DOI: 10.1016/j.ijmecsci.2020.105460.
11. Galileev, S. M., Matrosov A. V. Method of initial functions: Stable algorithms in the analysis of thick laminated composite structures. Composite Structures, 1997, no. 39 (3—4), pp. 255—262. DOI: 10.1016/S0263-8223(97)00108-6.
12. Goloskokov D. P., Matrosov A. V. Approximate analytical approach in analyzing an orthotropic rectangular plate with a crack. *Materials Physics and Mechanics*, 2018, vol. 36 (1), pp. 137—141. DOI: 10.18720/MPM.3612018_15.
13. Hutchinson R. G., Fleck N. A. The structural performance of the periodic truss. *Journal of the Mechanics and Physics of Solids*, 2006, vol. 54, no. 4, pp. 756—782. DOI: 10.1016/j.jmps.2005.10.008.
14. Hutchinson R. G., Fleck N. A. Microarchitectured cellular solids — the hunt for statically determinate periodic trusses. *ZAMM Z. Angew. Math. Mech*., 2005, vol. 85, no. 9, pp. 607—617. DOI: 10.1002/zamm.200410208.
15. Kazmiruk I. Yu. On the arch truss deformation under the action of lateral load. *Science Almanac*, 2016, no. 3—3 (17), pp. 75—78. DOI: 10.17117/na.2016.03.03.075.
16. Kirsanov M. N. Deformations and Spatial Structure Vibrations Frequency of the Rectangular Contour Type Cover: Analytical Solutions. *Construction of Unique Buildings and Structures*, 2021. 98, article No 9805. DOI: 10.4123/CUBS.98.5.
17. Kirsanov M. N. *Planar Trusses: Schemes and Formulas*. Cambridge Scholars Publishing. Lady Stephenson Library, Newcastle upon Tyne, NE6 2PA, UK, 2019. 206 p.
18. Kirsanov M. N. *Trussed Frames and Arches: Schemes and Formulas*. Cambridge Scholars Publishing Lady Stephenson Library. Newcastle upon Tyne, NE6 2PA, UK, 2020. 186 p.
19. Mathieson C., Roy K., Clifton G., Ahmadi A., Lim J. B. P. Failure mechanism and bearing capacity of cold-formed steel trusses with HRC connectors. *Engineering Structures*, 2019, vol. 201, 109741. DOI: 10.1016/j.engstruct.2019.109741.
20. Matrosov A. V. Computational Peculiarities of the Method of Initial Functions. *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, 2019, 11619 LNCS, pp. 37—51. DOI: 10.1007/978-3-030-24289-3_4.
21. Rakhmatulina A. R., Smirnova A. A. Analytical calculation and analysis of planar springel truss. *Structural mechanics and structures*, 2018, no. 2 (17), pp. 72—79.
22. Rakhmatulina A. R., Smirnova A. A. The dependence of the deflection of the arched truss loaded on the upper belt, on the number of panels. *Nauchnyi al'manakh*, 2017, no. 2—3 (28), pp. 268—271.
23. Vatin N. I., Havula J., Martikainen L., Sinelnikov A. S., Orlova A. V., Salamakhin S. V. Thin-walled cross-sections and their joints: tests and fem-modelling. *Advanced Materials Research*, 2014, vol. 945—949, pp. 1211—1215. DOI: 10.4028/www.scientific.net/AMR.945-949.1211.
24. Villegas L., Moran R., Garcia J. J. Combined culm-slat Guadua bamboo trusses. *Engineering Structures*, 2019, vol. 184, pp. 495—504. DOI: 10.1016/j.engstruct.2019.01.114.
25. Voropai R. A., Kazmiruk I. Yu. Analytical study of the horizontal stiffness of the flat statically determinate arch truss. *Bulletin of Scientific Conferences*, 2016, no. 2—1 (6), pp. 10—12.
26. Zok, F. W., Latture R. M., Begley M. R. Periodic truss structures. *Journal of the Mechanics and Physics of Solids*, 2016, vol. 96, pp. 184—203. DOI: 10.1016/j.jmps.2016.07.007.