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Dependence of the Two-Span Truss Bridge Vibration Frequency on the Number of Panels

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Abstract:

The object of the research is a planar, externally statically indeterminate truss with a cross-shaped lattice. The truss has supports at the ends and in the middle. The dependence of the lowest frequency of vibrations of the truss is found under the assumption that the mass of the structure is concentrated in its nodes. Both horizontal and vertical displacements of nodes are taken into account. **Method.** The reactions of the supports and the forces in the rods are found in an analytical form by the method of cutting nodes in the Maple computer mathematics system. The stiffness matrix is calculated using the Maxwell-Mohr formula. The results of calculating the first natural frequency by the Dunkerley method of a series of solutions for trusses with a different number of panels are generalized by induction to an arbitrary number of panels. **Results**. A comparison of the analytical expression for the first frequency with the lowest value of the natural oscillation spectrum obtained numerically shows the high accuracy of the derived formula. It is noted that with an increase in the number of panels, the accuracy of the approximate analytical solution increases, reaching several percent with the number of panels in each span of more than twenty.

1 Introduction

One of the main characteristics of the dynamics of the structure is its first (lowest) natural frequency. The calculation of the natural frequencies of structures is of particular importance for their seismic safety [1]–[4]. If the construction has many degrees of freedom, then the calculation of the spectrum of natural oscillations, as a rule, is reduced to using a numerical method based on the finite element method [5]–[10]. Analytical solutions obtained in computer mathematics systems are a good control of the accuracy of a numerical solution. For regular systems with periodically repeating constructive parts, an inductive method of generalizing partial solutions to an arbitrary order of regularity is used. Thus, for example, deflections of some schemes of planar [11]–[14] and spatial [15] trusses are found depending on the number of panels. The calculation of the natural oscillation frequencies in the analytical form in the Maple [16], [17] system is obtained in [18]–[20]. Analytical solutions are especially important for evaluating numerical solutions for systems with a large number of elements, in the calculations of which errors of accumulated rounding and a sharp increase in counting time are especially evident. General questions of the existence and calculation of statically determinate regular systems are considered in [21]–[23]. In the monograph [24], algorithms for calculating both the statics and dynamics of regular trusses, including statically indeterminate ones, are given, the concept of biregularity of the structure (truss, cross beam system, etc.) is introduced. The mathematical basis used in this work is the difference calculus. In [25], the existence of spectral constants (frequencies that are the same for regular trusses of different orders) and spectral isolines for a spatial regular cantilever truss is shown. Nonlinear problems of calculating trusses were solved in the numerical form [26]–[29]. Algorithms for calculating regular rod systems and in the system of computer mathematics are given in [30]. Methods for analyzing the deformation of rod elastic systems of a regular structure are studied in [31]. Small elastic vibrations of plane trusses of an orthogonal structure are considered in [32].

2 Materials and Methods

2.1 Truss scheme. Calculation of the compliance matrix

Let's consider a planar model of a truss of a two-span continuous bridge. The truss with a cross-shaped grid has two movable and one fixed hinge support (Fig. 1) and contains $\mu = 8n+4$ members, where n is the number of panels in one span. The length of each span is $a(2n-1)/2$. The construction is statically determinate, but the four reactions of the supports cannot be found from the truss equilibrium equations. The additional support creates an external static indeterminacy. The reactions of the supports are obtained together with the forces in the rods from the system of algebraic equations of equilibrium of the nodes. To obtain an analytical solution, the symbolic mathematics of Maple and the program [33] is used. The coordinates of the nodes and the order of connecting the rods are entered into the program, just as graphs are introduced in discrete mathematics [7]. The matrix of the system of equilibrium equations consists of the guiding cosines of the forces in the rods, for the calculation of which information about the structure of the connections of the rods and the coordinates of the nodes are used. The mass of the truss is conditionally distributed evenly across all $N_0 = 4n-1$ nodes, not including the three support nodes here. Given that each such node has two degrees of freedom, the system under consideration has $N = 2N_0$ degrees of freedom in this formulation.

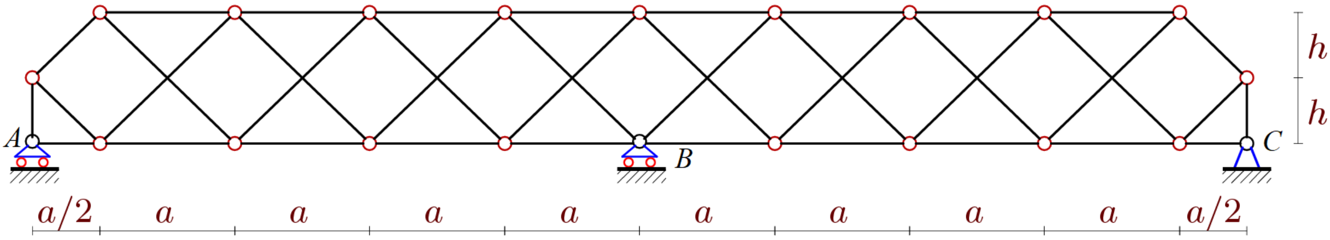

Fig. 1 – Truss, *n=5*

We will write the system of equations of cargo movement in matrix form

$$\mathbf{M}_N \ddot{\mathbf{U}} + \mathbf{D}_N \mathbf{U} = 0, \tag{1}$$

where $\mathbf{U} = [u_1, ..., u_N]^T$ – displacements of masses, $\mathbf{D}_N$ – stiffness matrix, $\mathbf{M}_N$ – inertia matrix, $\ddot{\mathbf{U}}$ – acceleration vector. In the case of identical masses, the inertia matrix is proportional to the unit matrix $\mathbf{M}_N = m\mathbf{I}_N$. The stiffness matrix $\mathbf{D}_N$ can be calculated as the inverse matrix of the compliance matrix $\mathbf{B}_N$, the elements of which are found by the Maxwell-Mohr formula

$$b_{i,j} = \sum_{\alpha=1}^{\mu-4} S_\alpha^{(i)} S_\alpha^{(j)} l_\alpha / (EF). \tag{2}$$

Here, the summation is carried out for all the rods, except for the four supports ones, which are conditionally rigid. The designations are introduced: $S_\alpha^{(i)}$ is the force in the member from the action of a single force at node i in the direction of vibrations, EF is the stiffness of the rods, l_α is the length of the member α. To determine the forces, the program [33] is used in the Maple language [16].

If we multiply (1) by the matrix $\mathbf{B}_N$, then for harmonic oscillations of the form

$$u_k = \Phi_k \sin(\omega t + \varphi_0) \tag{3}$$

with amplitudes Φ_k, there is a replacement $\ddot{\mathbf{U}} = -\omega^2 \mathbf{U}$. The problem is reduced to the problem of eigenvalues for the matrix $\mathbf{B}_N$: $\mathbf{B}_N \mathbf{Y} = \lambda \mathbf{Y}$, where ω is the natural frequency of oscillations, $\lambda = 1/(m\omega^2)$ is the eigenvalue of the matrix $\mathbf{B}_N$. In general, such a problem can only be solved numerically.

2.2 Dunkerley's method

We obtain the value of the first frequency using the approximate Dunkerley method [34], which gives the lower bound of this value. The partial frequencies included in the solution according to the Dunkerley method are divided into oscillation frequencies along the x - axis $\omega_{k,x}$ and the y - axis $\omega_{k,y}$. The index k means the number of the node with mass m. The solution takes the form:

$$\omega_D^{-2} = \sum_{k=1}^{N_0} (\omega_{k,x}^{-2} + \omega_{k,y}^{-2}). \tag{4}$$

Equation (1) in the case of oscillations of one mass has a simple scalar form:

$$m\ddot{u}_k + d_k u_k = 0,\ k = 1,...,N, \tag{5}$$

where u_k is the displacement of the mass, $\ddot{u}_k$ is the acceleration, d_k is the stiffness coefficient. The frequency of vibration of the load is $\omega_k = \sqrt{d_k / m}$. The stiffness coefficient is determined by the Maxwell-Mohr formula (2), which in this case has a simplified form:

$$\delta_k = 1/d_k = \sum_{\alpha=1}^{\mu-4} \left(S_\alpha^{(k)}\right)^2 l_\alpha / (EF). \tag{6}$$

Here it is indicated $S_\alpha^{(k)}$ — the forces in the rod are numbered α from the action of a unit vertical force applied to the node where the mass numbered k is located. According to (4) we have

$$\omega_D^{-2} = m\sum_{k=1}^{N_0} \delta_k = m(\Delta_x + \Delta_y). \tag{7}$$

The sums Δ_x and Δ_y are calculated separately. The calculation of the compliance δ_k , $k = 1,..., N_0$ from the action of vertical unit forces on the nodes and summing them for a different number of panels shows that the overall appearance of the sum does not change

$$\boxed{\Delta_{y,n} = (C_1 a^3 + C_2 c^3 + C_3 h^3) / ((2n-1)^2 h^2 EF).} \tag{8}$$

The length of the brace is indicated $c = \sqrt{a^2 + 4h^2}$. As a result of calculations, we obtain the following sequence of solutions:

$$\begin{aligned}
&\Delta_{y,2} = 5(40a^3 + 21c^3 + 128h^3) / (72h^2 EF),\\
&\Delta_{y,3} = (824a^3 + 165c^3 + 544h^3) / (40h^2 EF),\\
&\Delta_{y,4} = (4368a^3 + 455c^3 + 1024h^3) / (56h^2 EF),\\
&\Delta_{y,5} = (45744a^3 + 2907c^3 + 4960h^3) / (216h^2 EF),...
\end{aligned} \tag{9}$$

Using the induction method involving the *rgf_findrecur* and *rsolve* operators from the special *genfunc* package of the Maple system, we obtain common terms of sequences. In this problem, the minimum number of trusses with a consistently increasing number of panels required to find the regularity of the formation of coefficients is 10. For coefficients at powers a^3, c^3 , and h^3, we have the following power-type expressions:

$$\begin{aligned}
&C_1 = n(n-1)(2n-1)^2(46n^2 - 46n + 33) / 90,\\
&C_2 = (4n-1)(4n-3)(2n-1)^2 / 24,\\
&C_3 = 4n(2n-1)(7n-4) / 3.
\end{aligned} \tag{10}$$

Similarly, when calculating the partial frequencies of mass oscillations horizontally, we have a general view of the corresponding sum

$$\Delta_{x,n} = (C_4 a^3 + C_5 c^3 + C_6 h^3) / (2(2n-1)^2 a^2 EF). \tag{11}$$

The coefficients in this expression have the form

$$C_4 = (2n-1)(8n^4 + 32n^3 - 42n^2 + 2(3(-1)^n + 7)n - 3(-1)^n)/3,$$
$$C_5 = (2n-1)(6n^2 - 4n + 1), \tag{12}$$
$$C_6 = 32n^2.$$

3 Results and Discussion

Formulas (7), (8), (10-12) give the following lower estimate of the first frequency:

$$\omega_D = (2n-1)\sqrt{\frac{EF}{m\left((C_1a^3 + C_2c^3 + C_3h^3)/h^2 + (C_4a^3 + C_5c^3 + C_6h^3)/(2a^2)\right)}}. \tag{13}$$

The accuracy of the obtained estimate can be determined by comparing the entire natural frequency spectrum of the structure with the first frequency. For the numerical solution of this problem, the Maple system has the *Eigenvalues* operator. Figure 2 compares the first frequency obtained by the formula (13) and the lowest frequency of the entire frequency spectrum. The following design parameters are taken: $E = 2 \cdot 10^5$ MPa — the elastic modulus of the rod material, $F = 2.0$ sm^2 — the cross-sections of the rods, $m =$ 300 kg. The following dimensions are taken: $a =$ 4.0 m, h = 3.0 m. With an increase in the number of panels, the dependence curve was obtained numerically and the curve was constructed according to the formula (13) approach.

We introduce the value of the relative error $\varepsilon = (\omega_1 - \omega_D)/\omega_1$. Figure 3 shows that the error of the analytical solution is small, but with an increase in the number of panels, it decreases unevenly.

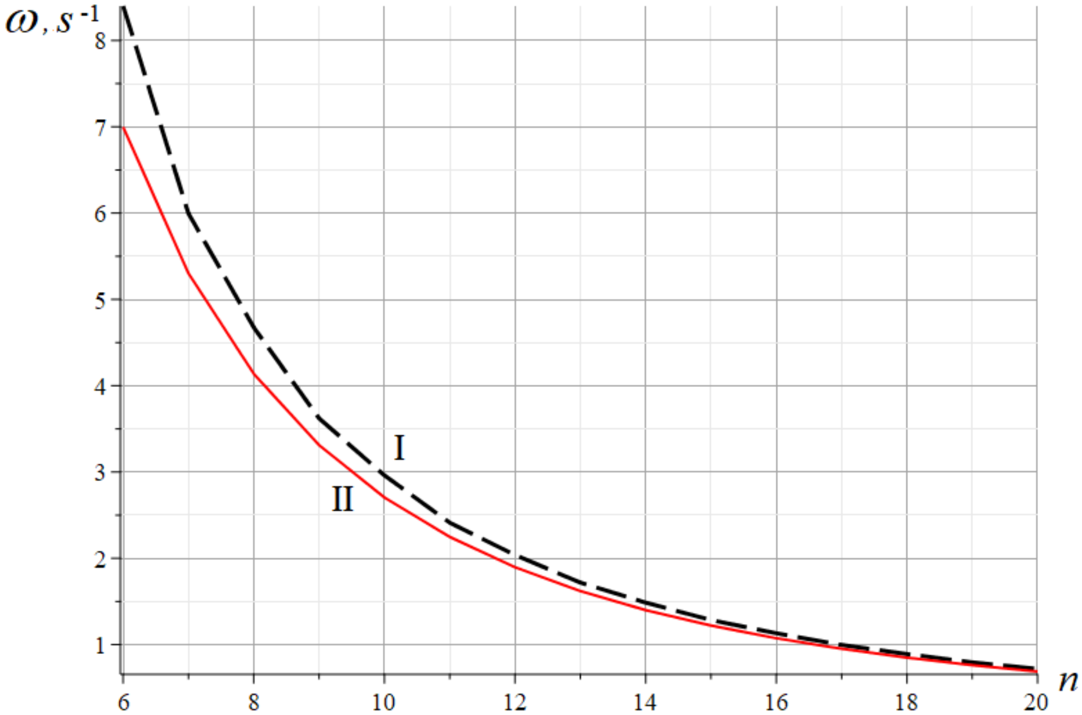

Fig. 2 – Frequency dependence on the number of panels,
I – numerical solution; II – analytical assessment

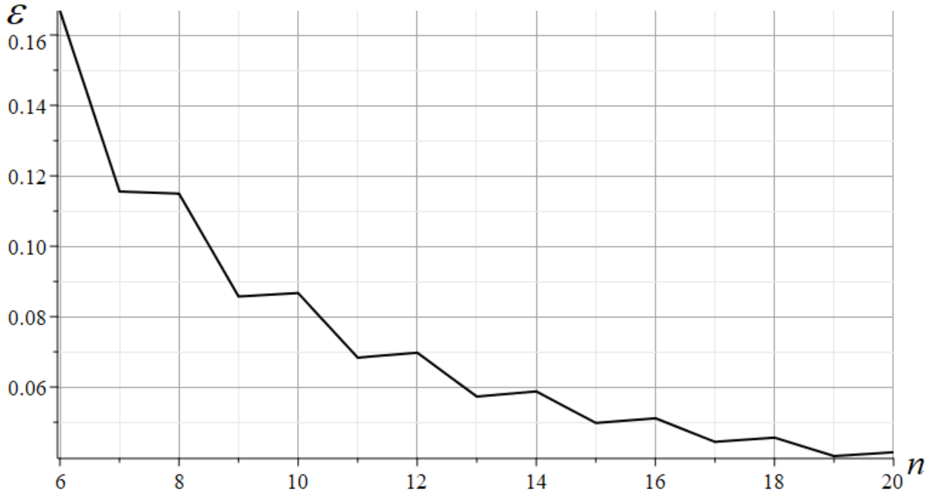

Fig. 3 – Dunkerley's estimation error depending on the number of panels

The spectra of trusses with a different number of panels and with the same parameters as in the graphs constructed above are shown in Figure 4. Each curve connecting the points corresponding to the natural frequencies of vibrations corresponds to one truss of order n. The frequency numbers in the spectrum are plotted on the abscissa axis. Compared with the results [25] obtained for the spatial console, the spectra do not have the same regular structure. The family of spectra does not contain isolines and spectral constants. The frequencies are arranged in the spectra almost randomly. However, even here we can notice an implicitly expressed upper bound of frequencies and a lower inclined bound with an angle of inclination decreasing with the growth of the largest order of the considered set of regular trusses. The absence of a regular structure of the family of spectra in this problem can be explained by the fact that the spectra contain the oscillation frequencies of individual masses both vertically and horizontally. These frequencies have a different nature, which can be seen by comparing formulas (8) and (11). In one case, the denominator of the sum is the vertical size, in the other — horizontally. The horizontal and vertical stiffness of the truss is significantly different. In [25] the problem statement was simpler. The masses in the nodes of the spatial console moved only vertically, horizontal mass displacements were not taken into account

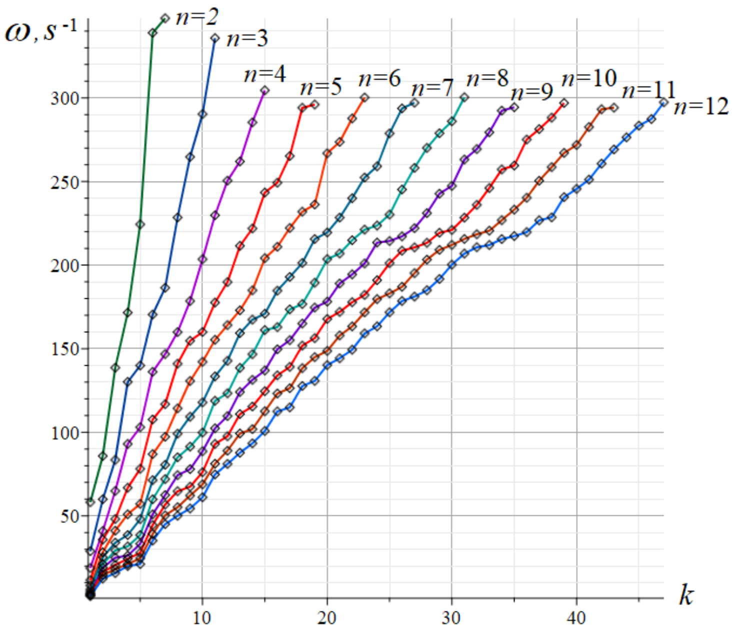

Fig. 4 – Frequency spectra of a family of regular trusses

The study of frequency patterns in the spectra and analytical expressions for the first frequency can be used in optimization problems of truss structures [4], [35]–[37]. The analytical nature of the solution simplifies its use, and the observed increase in its accuracy with a large number of panels, i.e. just in cases when numerical solutions are especially unreliable, makes such solutions attractive.

4 Conclusions

The main results of the work are as follows.

1. A formula for the dependence of the first oscillation frequency of a two-span truss with an arbitrary number of panels is obtained.

2. It is shown that the accuracy of the found estimate increases with an increase in the number of panels.

3. The picture of a family of regular truss spectra of various orders is analyzed. It is noted that there is an upper bound for all natural frequencies.

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