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Deflection and First Natural Frequencies of a Regular Arched Truss Oscillation

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Abstract—The dependences of the first four frequencies of natural oscillations of a planar regular truss of the thrust type are obtained numerically. A model is used in which the mass of the truss is concentrated in its nodes. The Maxwell—Mohr formula is used to calculate the rigidity of the truss. For the first frequency, an analytical dependence on the number of panels is derived by the induction method using a simplified version of the Dunkerley method in the Maple computer mathematics system. Good agreement with the numerical result is shown. An analytical dependence of the static deflection of the truss on its dimensions and load is obtained.

Keywords: Planar truss, natural oscillations, first oscillation frequencies, deflection, Dunkerley method, analytical solution, Maple, fundamental frequency, Maxwell—Mohr formula, regular truss

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INTRODUCTION

In practical calculations of the natural vibration frequency of structures, specialized numerical packages based on the finite element method are usually used [1–3]. An alternative calculation method is analytical, applicable to statically determinate regular trusses. Two simple methods are known that provide estimates of the first frequency: the Dunkerley method (lower estimate) and the Rayleigh method (upper estimate) [4]. Here, formulas are obtained for a two-sided estimate of the vibration frequency of a planar cantilever truss with a diagonal lattice. In [5], a simplified version of the Dunkerley method with a more accurate analytical solution is given. The analytical solution in the form of a finite formula can be used to estimate the numerical solution, especially since the accuracy of such a method is not related to the number of rods in the structure, while the finite element method for large-scale systems is prone to error accumulation. In [6], an analytical estimate of the fundamental natural vibration frequency of a regular lattice truss is obtained and the spectrum of all frequencies is analyzed. An estimate of the fundamental frequency of oscillations of a spatial regular truss with a horizontal beam in the form of a compact formula is given in [7]. In [8, 9], the dependence of the region of resonance-safe frequencies of the spectrum of natural oscillations of a flat regular truss with an arbitrary number of panels on the problem parameters is investigated. An analytical solution to the problem of the frequency of oscillations of a spatial cantilever truss is constructed in [10]. A formula for the natural frequency of oscillations of a planar regular truss is obtained in [11]. The reference book [12] contains diagrams of planar regular trusses and formulas for calculating their deflections, forces in characteristic rods and support displacements.

THE TRUSS SCHEME

Let us consider the scheme of a statically determinate truss with parallel chords (Fig. 1). The truss has a cross-shaped lattice and two fixed supports. The middle part, raised to a height h, contains $2n$ panels of length $2a$ and height $2h$.

The truss consists of rods and nodes. The number of rods does not include four rods simulating two fixed supports.

To calculate the rigidity of the structure using the Maxwell-Mohr formula when determining the vibration frequencies by the method of cutting out nodes, the forces in the rods are found. The scheme of the structure is specified by the coordinates of the nodes and the order of connecting the rods into nodes. The origin of coordinates is located at the left support A (Figs. 1 and 2). The coordinates are as follows:

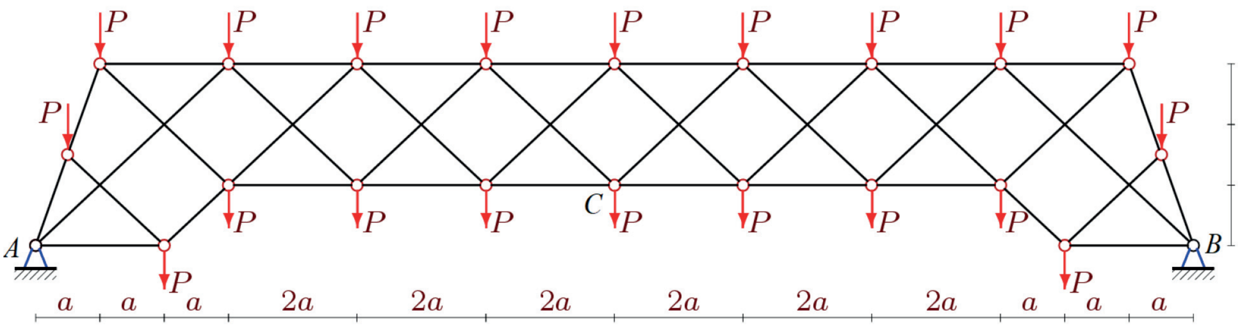

Fig. 1. Truss under the load, $n = 3$.

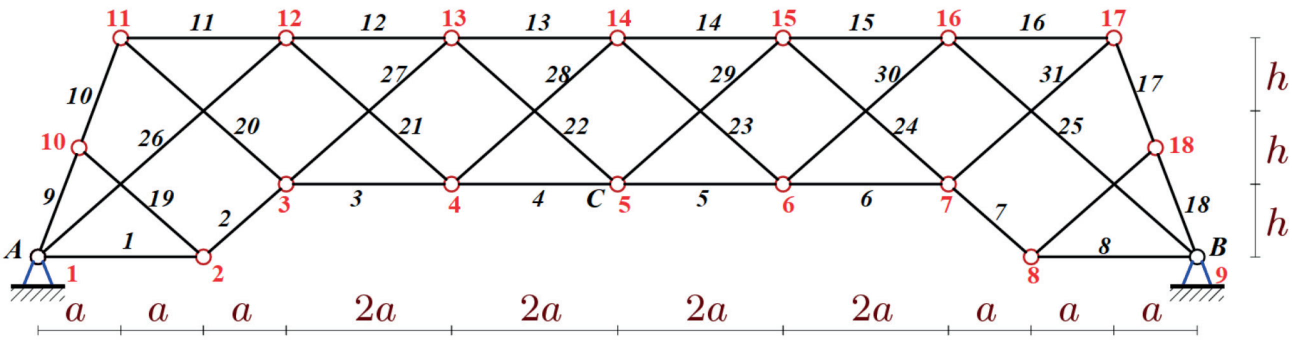

Fig. 2. Numbering of nodes and rods of the truss, $n = 2$.

$$
\begin{aligned}
&x_1 = 0,\quad y_1 = 0,\quad x_2 = 2a,\quad y_2 = 0,\\
&x_{i+2} = (2i+1)a,\quad y_{i+2} = h,\quad i = 1,..,2n+1,\\
&x_{i+2n+6} = (2i-1)a,\quad y_{i+2n+6} = 3h,\quad i = 1,..,2n+3,\\
&x_{2n+4} = L_0 - 2a,\quad y_{2n+4} = 0,\\
&x_{2n+5} = L_0,\quad y_{2n+5} = 0,\\
&x_{2n+6} = a/2,\quad y_{2n+6} = 3h/2,\\
&x_{2n+7} = a,\quad y_{2n+7} = y_{4n+10} = 3h/2,\\
&x_{4n+10} = L_0 - a/2.
\end{aligned}
$$

The order of connecting the rods into the nodes of the truss lattice is determined by special lists of node numbers at the ends of individual rods of the structure. The rods of the chords, for example, are specified by unoriented lists:

$$
\begin{gathered}
\Phi_i = [i, i+1],\\
\Phi_{i+2n+5} = [i+2n+5, i+2n+6],\\
i = 1,..,2n+4.
\end{gathered}
$$

The condition of equilibrium of nodes is written in the form of equations in projection on the coordinate axes. The coefficients of these equations are the direction cosines of the forces:

$$
\begin{gathered}
l_{x,i} = (x_{\Phi_{i,1}} - x_{\Phi_{i,2}})/l_i,\\
l_{y,i} = (y_{\Phi_{i,1}} - y_{\Phi_{i,2}})/l_i,\quad i = 1,\ldots,\eta,
\end{gathered}
$$

where $l_i = \sqrt{l_{x,i}^2 + l_{y,i}^2}$. Matrix G of the system of linear equations of equilibrium of nodes is formed as follows:

$$
\begin{gathered}
G_{2\Phi_{i,1}-1,i} = l_{x,i}/l_i,\quad G_{2\Phi_{i,1},i} = l_{y,i}/l_i,\\
G_{2\Phi_{i,2}-1,i} = -l_{x,i}/l_i,\quad G_{2\Phi_{i,2},i} = -l_{y,i}/l_i.
\end{gathered}
$$

The system of equations of equilibrium of nodes is written in matrix form:

$$
\mathbf{GS} = \mathbf{T}, \tag{1}
$$

where **T** is the vector of external nodal loads, **S** is the vector of efforts in rods. The efforts are found by solving a system of equations using the inverse matrix method in the Maple computer mathematics system.

DEFLECTION

The deflection of a truss with n panels in half the span is calculated based on the vertical displacement of the middle node of the lower chord using the Maxwell-Mohr formula:

$$\Delta_n = \sum_{i=1}^{\eta} S_i^{(P)} S_i^{(1)} l_i \Big/ (EF),$$

where $S_i^{(P)}$—the force in the rod with the number i from the action of the load P distributed over all nodes of the truss, $S_i^{(1)}$ is the force from a single force applied to the node C with the number $n+3$, the displacement of which is calculated, EF is the rigidity of the truss rods for longitudinal forces. The analytical dependence of the deflection on the number of panels is determined by the induction method by generalizing the sequence of solutions for trusses of different orders. The solution of system (1) in the Maple system gives the following sequence:

$$\Delta_1 = P(51a^3 + 11c^3 + 12h^3)\Big/(2h^2 EF),$$
$$\Delta_2 = P(12644a^3 + 528c^3 + 41d^3 + 576h^3)\Big/(72h^2 EF),$$
$$\Delta_3 = P(2273a^3 + 117c^3 + 60h^3)\Big/(6h^2 EF),$$
$$\Delta_4 = P(27276a^3 + 660c^3 + 19d^3 + 288h^3)\Big/(24h^2 EF),$$
$$\Delta_5 = P(11537a^3 + 249c^3 + 84h^3)\Big/(6h^2 EF), \ldots$$

To determine the general term of this sequence, it was necessary to extend it to 18 terms. The general form of the obtained solution is as follows:

$$\Delta_n = P(C_1 a^3 + C_2 c^3 + C_3 d^3 + C_4 h^3)\Big/(h^2 EF). \quad (2)$$

The Maple system operators from the solution of recurrent equations give the following coefficients:

$$C_1 = (60n^4 + 8(4(-1)^n + 25)n^3 + 2(32(-1)^n + 215)n^2 + 6(27(-1)^n + 79)n + 123(-1)^n + 135)/36,$$
$$C_2 = (2n^2 + (5 - (-1)^n)n + 2(-1)^n + 5)/2,$$
$$C_3 = (25 + 8n)((-1)^n + 1)/144,$$
$$C_4 = 2n + 4.$$

Dependence (2) has an asymptotic behavior that is cubic in the number of panels:

$$\lim_{n\to\infty} \Delta_n \Big/ n^3 = 5P_0 a^3 \Big/ (12h^2 EF),$$

where $P_0 = 4(n+2)P$ is the total load on the truss.

NUMERICAL DETERMINATION OF THE NATURAL OSCILLATIONS FREQUENCIES

The truss model assumes that the truss mass is uniformly distributed over the nodes by concentrated masses μ. Oscillations occur along the y axis. Horizontal displacements of masses are not taken into account, the number of degrees of freedom of the mass system is equal to the number of nodes $K = 4n + 10$. The equations of mass motion at the truss nodes are written in matrix form:

$$\mu I_K \ddot{Y} + D_K Y = 0. \quad (3)$$

Here Y is the vector of vertical displacements of the truss nodes, $\ddot{Y}$ is the acceleration vector, D_K is the identity matrix, I_K is the stiffness matrix. Assuming that the vibrations are harmonic with a frequency , the substitution $\ddot{Y} = -\omega^2 Y$ is valid. When multiplying Eq. (3) on the left by the compliance matrix B_K, the problem is reduced to the problem of the eigenvalues of the matrix B_K: $B_K Y = \lambda Y$, where $\lambda = 1\big/(\omega^2\mu)$ are the eigenvalues. The compliance matrix is the inverse of the stiffness matrix: $B_K = 1\big/D_K$. The values of the elements of this matrix are calculated using the Maxwell–Mohr formula:

$$b_{i,j} = \sum_{\alpha=1}^{\eta} S_\alpha^{(i)} S_\alpha^{(j)} l_\alpha \Big/ (EF), \quad (4)$$

where $S_\alpha^{(i)}$ is the force in the rod with number $\alpha = 1,..,\eta$ from the action of a single vertical force applied to node i. The number of rods includes four rods simulating fixed hinged supports. The length of the vertical support rods is taken to be h, and the horizontal ones are taken to be a. These lengths determine the rigidity of the supports. The rigidity EF of all truss rods is the same. The eigenvalues of the matrix for calculating the frequency spectrum can be calculated numerically in the Maple system.

Figure 3 shows the results of calculating the first three eigenfrequencies depending on the number of panels in the truss. The truss dimensions are: $h = 2$ m, $a = 3$ m, mass at nodes $\mu = 200$ kg, elastic modulus $E = 2.1\times10^5$ MPa, cross-sectional area of the rods $F = 9$ cm^2. The distribution density of various frequencies depends significantly on the number of panels.

For $n = 6$, for example, the two upper frequencies ω_3 and ω_4 coincide, and for a truss with one panel in half the span ($n = 1$), frequencies ω_1 and ω_2 coincide. An approximate analytical expression can be found for the first frequency.

FORMULA FOR FUNDAMENTAL FREQUENCY

For an approximate estimate of the lower limit of the fundamental frequency, the Dunkerley formula is known

$$\omega_D^{-2} = \sum_{j=1}^{K} \omega_j^{-2}, \quad (5)$$

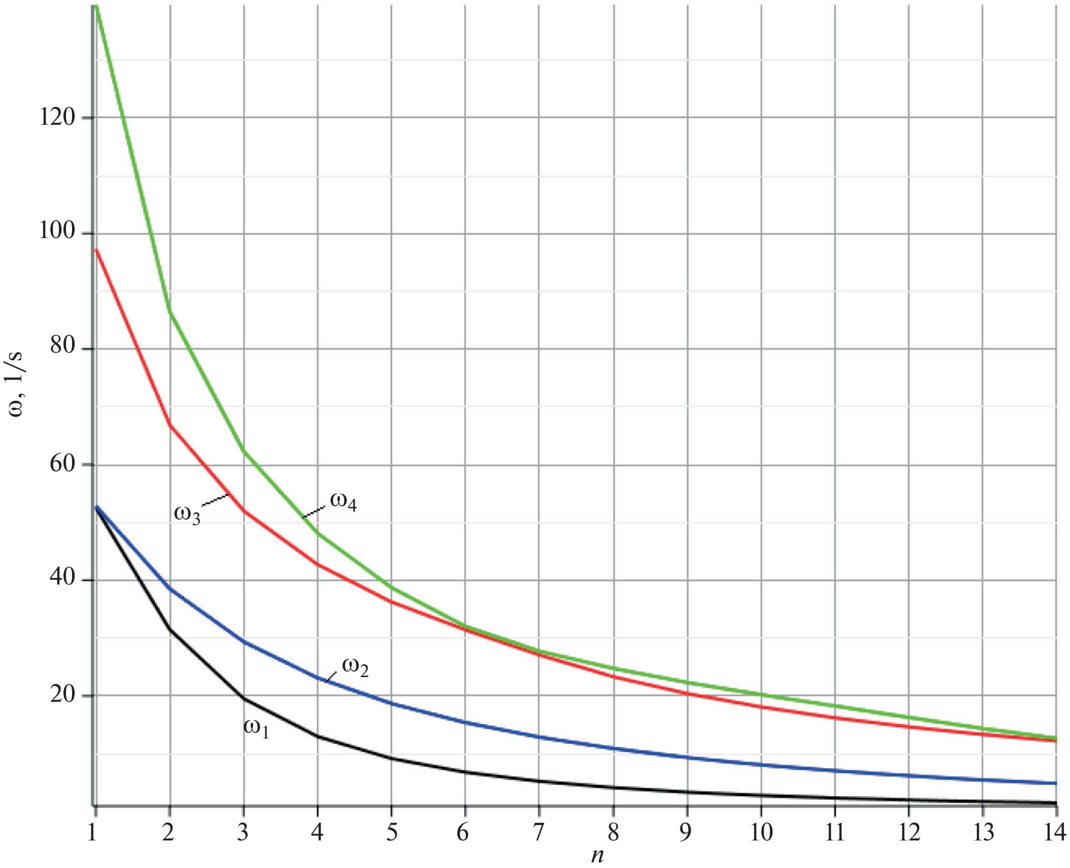

Fig. 3. Oscillation frequencies depending on the number of panels.

where ω_j is the partial frequency of the load at node j, calculated from the equation of its motion:

$$\mu\ddot{y}_j + D_j y_j = 0, \quad j = 1, 2, \ldots, K.$$

The stiffness coefficient D_j is the reciprocal of the compliance, which is calculated using the Maxwell-Mohr formula:

$$\delta_j = 1/D_j = \sum_{\alpha=1}^{\eta} \left(S_\alpha^{(j)}\right)^2 l_\alpha/(EF). \tag{6}$$

From (5) and (6) follows the formula for the lower limit of the first natural frequency according to Dunkerley:

$$\omega_D^{-2} = \mu\sum_{j=1}^{K} \delta_j = \mu\Delta_n. \tag{7}$$

The Dunkerley method has two disadvantages: an underestimated frequency value and the complexity of calculating the sum in symbolic form. A simplified version of the Dunkerley method [5] is free of these disadvantages, in which it is proposed to replace the sum with its approximate value calculated using the mean value theorem. The sum of the ordinates in (7) is associated with the area of a curvilinear figure, for the calculation of which the formula for the area of a triangle is used:

$$\omega_D^{-2} = \mu\sum_{j=1}^{K} \delta_j = \mu\delta_{\max} K/2 = \mu\Delta_{\max},$$

where $\delta_{\max}$ is the maximum value of δ_j. The point whose displacement under the force applied to it has the greatest value is selected empirically. In this problem, this is obviously the middle hinge C in the lower chord with number $n + 3$. Having calculated the value of the maximum deflection from a single force for trusses of different orders, we obtain the sequence

$$\Delta_{\max 1} = K(9a^3 + 5c^3 + h^3)/(4h^2EF),$$
$$\Delta_{\max 2} = K(409a^3 + 54c^3 + d^3 + 9h^3)/(36h^2EF),$$
$$\Delta_{\max 3} = K(65a^3 + 9c^3 + h^3)/(4h^2EF),$$
$$\Delta_{\max 4} = K(1697a^3 + 90c^3 + d^3 + 9h^3)/(36h^2EF),$$
$$\Delta_{\max 5} = K(233a^3 + 13c^3 + h^3)/(4h^2EF), \ldots$$

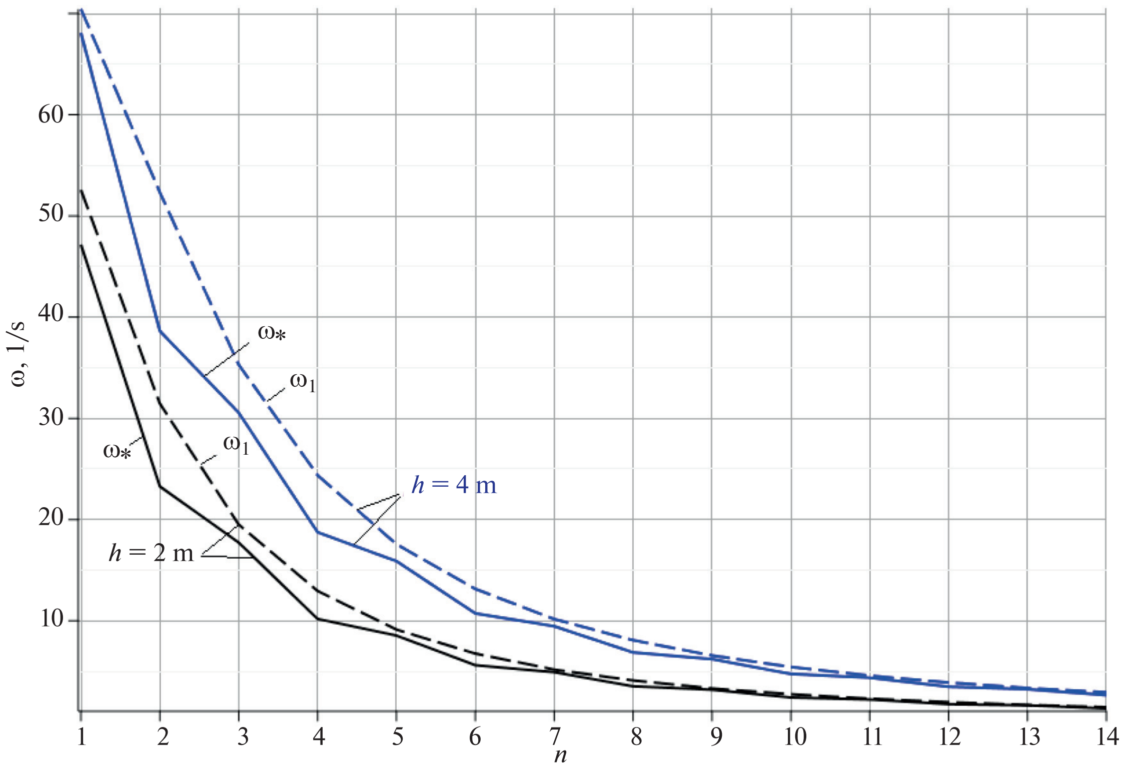

Fig. 4. Dependence of the first frequency of natural oscillations of the truss on the number of panels at $h = 2$ m and $h = 4$ m.

Generalizing this series to an arbitrary number of panels yields the formula

$$\Delta_{\max} = (4n+10) \times (C_1a^3 + C_2c^3 + C_3d^3 + h^3/4)/(h^2EF),$$

where

$$C_1 = (12n^3 + 6(2(-1)^n + 5)n^2 + 8(2(-1)^n + 5)n + 3(-1)^n + 30)/36,$$

$$C_2 = (4n - (-1)^n + 5)/8,$$

$$C_3 = ((-1)^n + 1)/72.$$

From this follows the formula for calculating the first frequency of free oscillations of the truss:

$$\omega_* = h\sqrt{\frac{EF}{\mu(4n+10)(C_1a^3 + C_2c^3 + C_3d^3 + h^3/4)}}. \quad (8)$$

COMPARISON OF SOLUTIONS. NUMERICAL CALCULATION

To estimate the approximate analytical solution (8), it is necessary to find the first frequency numerically. The same truss parameters are adopted as in the solution of the problem of the first four frequencies in Fig. 3. Figure 4 shows a comparison of the analytical dependence (8) of the frequency ω_* on the number of panels and the frequency ω_1 obtained numerically. Two variants of the height h are considered. With an increase in the number of panels, the natural frequency decreases monotonically, and the results of the analytical calculation approach the numerical one. To clarify the error of the analytical method, we introduce the relative value $\varepsilon_* = |\omega_1 - \omega_*|/\omega_1$. It is evident from Fig. 5 that the accuracy of formula (8) increases with an increase in the number of panels. For trusses with a smaller height, the error is slightly smaller. The parity of the number of panels also significantly affects the accuracy. For example, with $n = 5$, the accuracy is three times greater than with $n = 6$.

CONCLUSIONS

The paper considers a scheme of a planar arched truss. A formula for the dependence of the truss deflection under the action of a distributed nodal load is constructed and analyzed. The first four natural frequencies of oscillations are calculated numerically depending on the number of panels. An approximate analytical expression is obtained for the first frequency

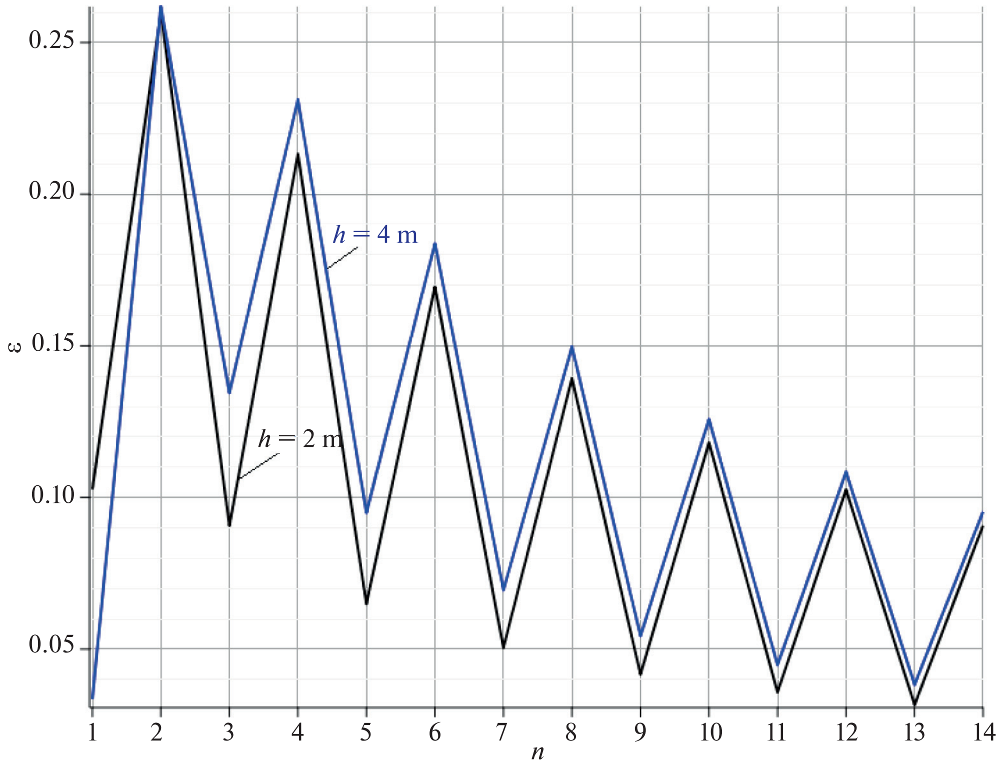

Fig. 5. Dependences of the error of the analytical solution on the number of panels.

using the induction method. It is shown that the accuracy of this solution increases with an increase in the number of panels. The proposed algorithm for constructing an analytical solution can be used to calculate the fundamental frequency of regular statically determinate trusses. One of the advantages of an analytical solution, in addition to its obvious simplicity, is that while the error of a numerical solution regularly increases with an increase in the number of panels, it decreases for an analytical solution. The resulting formula can serve as a simple estimate of a numerical solution obtained for a more accurate model of the same truss, for example, taking into account the masses of the rods.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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SPELL: 1. Dunkerley, 2. unoriented