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АНАЛИТИЧЕСКИЙ РАСЧЕТ ПРОГИБА СТЕРЖНЕВОЙ РАМЫ С ПРОИЗВОЛЬНЫМ ЧИСЛОМ ПАНЕЛЕЙ

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Предлагается модель статически определимой рамы-фермы и формулы для расчета ее прогибов под действием различных нагрузок. Модель определяется двумя натуральными параметрами — числом панелей в ригеле и числом панелей в вертикальных стойках - фермах. Выводятся формулы зависимостей усилий в наиболее опасных стержнях от нагрузок, числа панелей, упругих характеристик материала и размеров конструкции. Смещение узлов рассчитываются с помощью интегральной формулы Максвелла – Мора. Усилия в стержнях конструкции вычисляются в символьной форме с помощью системы символьной математики Maple. Для обобщения частных решений на произвольные числа панелей используется метод двойной индукции.

Ключевые слова: ферма, рама, индукция, аналитическое решение, Maple

For a long time, it was believed that the only way to assess the stress state of existing and designed structures (except for the experimental one) is numerical calculation [1-7]. With the advent of symbolic mathematics systems, the monopoly on computation in numerical methods gradually begins to disappear. Of course, such algorithms for symbolic mathematics have not yet been created that could obtain a compact formula for a complex structure taking into account all or most of its characteristics, but time goes on and the experience of programmers is growing. If at first systems like Reduce, Maple, and others could simply replace the numerical data in the condition of the problem and the equations used with symbols with a simple solution suitable for a particular design and specific load, recently, they have joined the number of free (variable) parameters and natural numbers characterizing the format of the design. In trusses, this is the number of panels [8-10], in frames, the number of storeys and the number of panels in the crossbar.

Consider a statically determinate frame consisting of $2n$ triangular panels in a crossbar and m panels with a cross-shaped lattice in side support trusses (Fig. 1).
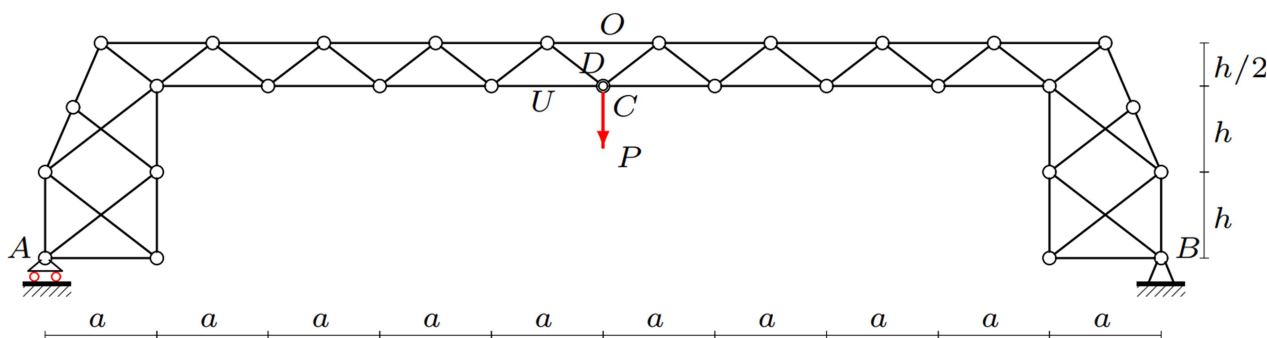

Fig. 1. Truss, concentrated load, n=4, m=2

The truss under study is statically determinate. The number of hinge assemblies $4(m+n)+5$, the number of rods, including three support rods modeling the left movable and right stationary hinge support, is equal $K = 8(m+n)+10$. A design feature is the beveled sides of the upper panels of the side parts. From the system of equations of equilibrium of nodes in projections on the axis, one can find the forces in the rods and the reactions of the supports.

Analytical calculations of the deflection and displacement of the movable support of similar structures in the Maple computer mathematics system were obtained in [11-17].

The solutions obtained for farms with a successively increasing number of panels were generalized to the general case by induction using special operators of the Maple system to compose and solve recurrence equations that are satisfied by the members of the sequences of coefficients in the desired formulas. The same method is used in the present work.

Deflection (vertical displacement of the middle node of the lower crossbar belt) is determined by the Maxwell-Mohr formula under the assumption that all the rods have the same longitudinal stiffness *EF*:

$$\Delta = \sum_{k=1}^{K-3} S_k^{(P)} S_k^{(1)} l_k / (EF),$$

where *E, F* are the elastic modulus and cross-sectional area of the rods, $S_k^{(1)}$ — is the force in the rod *k* from the action of a single vertical force in the middle node of the crossbar, $S_k^{(P)}$ — is the force in the rod *k* from the load, l_k is the length of the rod k. Three support rods are assumed to be rigid. Consider several types of loads.

1. Under the action of a concentrated force in the middle of the span, we have a general view of the deflection

$$\Delta = P(C_{1,n}a^3 + C_{2,n}c^3 + C_{3,n}h^3 + C_{4,n}d^3) / (EFh^2), \tag{1}$$

where $c = \sqrt{a^2 + h^2}$, $d = \sqrt{a^2 + 9h^2}$ is the length of the braces in the supporting parts.
Solving a series of problems for farms with a successively increasing number of panels n for a fixed $m = 1$, we obtain the following values for the coefficient $C_{1,n}$

21/2, 36, 171/2, 167, 577/2, 458, 1367/2, 973...

The **rgf_findrecur** operator of the **genfunc** package of the Maple system gives the following recursive equation

$$C_{1,n} = 4C_{1,n-1} - 6C_{1,n-2} + 4C_{1,n-3} - C_{1,n-4}.$$

Solving a homogeneous linear equation returns the **rsolve** operator

$$C_{1,n} = 4n^3/3 + 4n^2 + 25n/6 + 1.$$

Similarly, we obtain the coefficient values in (1) for calculating the deflection:

$$C_{2,n} = (8n+11)/16,$$
$$C_{3,n} = 0, \, C_{3,n} = 1/16.$$

To obtain the formula for the dependence of the deflection on the number *m*, it is necessary to obtain the coefficient values at *m* = 2,3,4, ... In this case, the task turned out to be simple: only the coefficient $C_{3,n} = (m-1)/2$ at h^3 depends on the number of vertical panels.

2. Under the action of a distributed load, the deflection of the middle span (Fig. 2) has the same form (1). Odds are also obtained by double induction

$$C_{1,n} = (n+1)(5n^3 + 15n^2 + 16n + 3)/3,$$
$$C_{2,n} = (4n^2 + 11n + 5)/8,$$
$$C_{3,n} = (n+1)(m-1), \quad C_{4,n} = (n+1)/8.$$

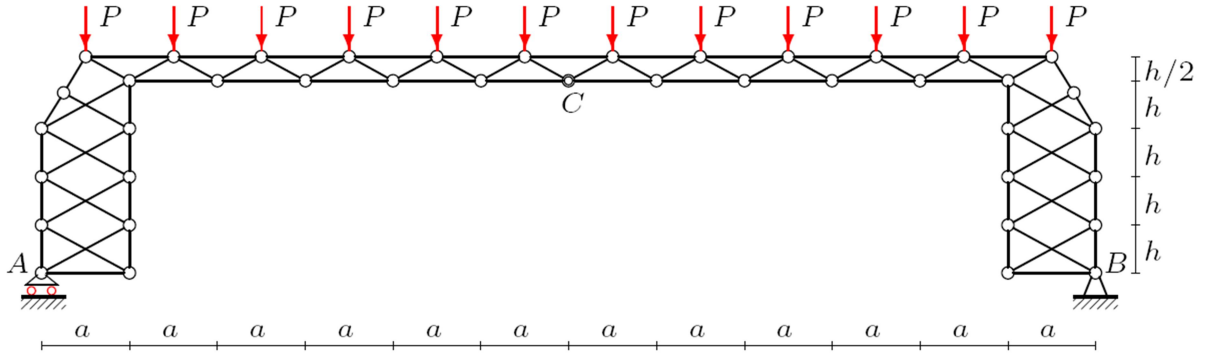**Fig. 2.** Truss, distributed load, $n=5$, $m=4$

3. The displacement of the movable support from the action of a concentrated force (Fig. 1) has the form

$$\delta_A = P(A_{1,n}a^3 + A_{2,n}c^3 + A_{3,n}h^3 + A_{4,n}d^3)/(ahEF), \qquad (2)$$

Where the coefficients are obtained by induction on the parameters m and n:

$$A_{1,n} = (1+4m)n^2 + (2+8m)n + 2m,$$
$$A_{2,n} = (11m+4)/8,$$
$$A_{3,n} = (1+2m^2 - 4m - (-1)^m)/4$$
$$A_{4,n} = m/8.$$

The displacement of the movable support from the action of a uniformly distributed load has the same form, but with coefficients

$$A_{1,n} = 2(n+1)(2n^2 + 8n^2m + 16nm + 4n + 3m)/3,$$
$$A_{2,n} = (11m+4)n + 5m + 4,$$
$$A_{3,n} = (n+1)(2m^2 - 4m + 1 - (-1)^m)/2,$$
$$A_{4,n} = m(n+1)/4.$$

4. The vertical displacement of the middle node C of the crossbar from the action of a horizontal lateral load uniformly distributed over the height (Fig. 3) has the form

$$\Delta = P(C_{1,n}a^3 + C_{2,n}c^3 + C_{3,n}h^3 + C_{4,n}d^3)/(ahEF),$$

where only the coefficient depends on the number n C_1

$$C_{1,n} = (1 + 4n + 2n^2)(1 + 2m^2 + 6m)/4,$$
$$C_{2,n} = (22m^2 + 66m + 5)/64,$$
$$C_{3,n} = (4m^3 + 3m^2 - 22m + 12 - 3(-1)^m)/24,$$
$$C_{4,n} = (m^2 + 3m - 1)/32.$$

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5. The horizontal displacement of the node C from the action of the horizontal force on the crossbar (Fig. 4) has the form

$$\delta_C = P(A_{1,n}a^3 + A_{2,n}c^3 + A_{3,n}h^3 + A_{4,n}d^3)/(a^2(n+1)^2 EF), \tag{3}$$

where

$$A_{1,n} = (2(4n+1)(8n^2+13n+6)m^2 + (26n^3+30n^2+n-6)m +$$
$$+ 3(n+1)(3n^2+2(-1)^m n + 4n+2+2(-1)^m))/12,$$
$$A_{2,n} = ((44n^2+60n+22)m^2 + (64n^2+109n+48)m-4n-4)/32,$$
$$A_{3,n} = (8(4n^2+5n+4)m^3 - (48n^2+54n+18)m^2 +$$
$$+ (16n^2+32n+12(-1)^m+12(-1)^m n+4)m+3(n+1)((-1)^m-1))/48,$$
$$A_{4,n} = m(4n^2m+4nm-n+2m)/32.$$

A feature of this solution is the dependence of the determinant of the system of equilibrium equations of nodes on the number of panels n, which manifests itself in the presence of a factor $(n+1)^2$ in the denominator (3). In all other solutions considered here, as well as in most known solutions [8-17], the determinant does not depend on the number of panels, which facilitates the determination of the common terms of the sequences.
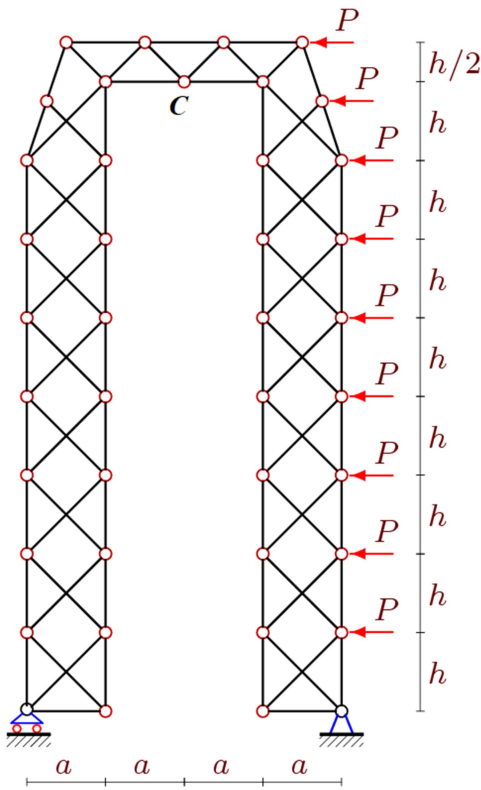

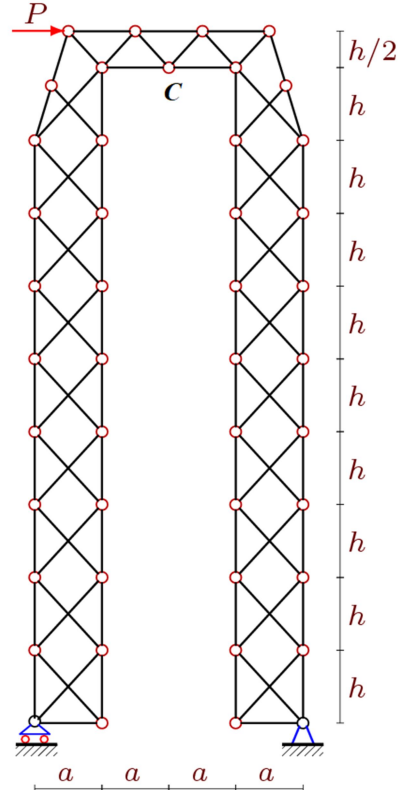**Fig. 3.** Frame under the action of lateral load, $n=1$, $m=8$

Fig. 4. Frame under the action of lateral load, $n=1$, $m=9$

6. To assess the strength and stability of the structure, it is useful to have formulas for the forces in the most compressed and extended rods. These formulas are automatically obtained when deriving formulas for deflection. It remains only to generalize to an arbitrary number of panels. For efforts in the brace and belts of the middle panel of the crossbar (Fig. 1) when loading the frame

with one force, we have the following expressions

$$D = Pc / (2h), \ O = -Pa(n+1) / h, \ U = Pa(2n+1) / (2h).$$

Note that these expressions do not depend on the number of panels m in the side racks and can be found quite simply by the Ritter section method. Similar expressions have efforts in case of loading with distributed load (Fig. 2):

$$D=0, \ O=-Pa(n+1)^2 / h, \ U=Pa(n+1)^2 / h.$$

Conclusion

The proposed scheme of a statically determinable frame with two supports allows simple analytical solutions for deflection and horizontal displacement under the influence of various loads. A linear combination of solutions makes it possible to obtain estimates of the deformability of the structure for a wide class of loads, and two independent natural parameters (n and m) make it possible to apply these estimates for a wide variety of structures — from frames of almost beam type ($m = 1$) to tower structures ($m >> n$). The applied calculation algorithm is easily tuned to arbitrary loads. Thus, a uniformly distributed lateral load simulating a wind load can be replaced by a load that increases linearly with height, a vertical uniformly distributed load can be replaced by some asymmetric model of crane equipment or some other temporary load. In addition, if we consider small deformations and linear formulation, all the proposed solutions will not change if hinges are placed at the points of intersection of the rods in the cruciform lattice. The static definability of the system will not change either, since each new node on the one hand gives two additional equations and, on the other hand, two additional rods appear in the design.

The algorithm used to derive analytical solutions is applicable for spatial trusses [18–20] and in other regular systems, in particular, in problems of structures from a cross-layered beam subjected to bending [21–22].

Reviews of analytical solutions for deflection of trusses with an arbitrary number of panels were made in [9,23].

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ANALYTICAL CALCULATION OF THE DEFLECTION OF THE ROD FRAME WITH AN ARBITRARY NUMBER OF PANELS

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A model of statically determinate frames-truss and formulas for the calculation of its deflection under the action of various loads proposed. The model is determined by two natural parameters — the number of panels in the crossbar and the number of panels in vertical racks-trusses. A formulas dependency effort in the most dangerous rods under the load, number of panels, the elastic characteristics of the material and the size of the structure are obtained. Node displacements are calculated using the Maxwell-Mohr integral formula. The forces in the rods of the construction are calculated in symbolic form using the Maple symbolic mathematics system. The double induction method is used to generalize partial solutions to arbitrary numbers of panels.

Keywords: truss, frame, induction, analytical solution, Maple.