## BUILDING MECHANICS

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# ANALYTICAL CALCULATION OF THE DEFLECTION OF A SPATIAL HINGE-ROD FRAME WITH AN ARBITRARY NUMBER OF PANELS

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**Statement of the problem.** The task is to obtain in symbolic form the dependence of the deflection of the proposed scheme of a statically definable spatial truss of a regular type on the number of panels under various loads, including the load from the truss plane. A truss has two independent parameters that define its proportions.
**Results.** For several types of loading according to the Maxwell - Mohr formula, analytical dependences of the deflections of the structure on the number of panels, load, and dimensions are derived. When generalizing a series of partial solutions with a given number of panels to an arbitrary number of panels, together with operators of the Maple computer mathematics system, the induction method is used. Asymptotic approximations of solutions are obtained.
**Conclusions.** The proposed model of a spatial frame with two independent numbers of panels that define the proportions of the structure allows an analytical solution of the problem of deflection under different types of loading. The derived formulas can be used as test formulas for evaluating approximate numerical solutions and for optimization problems.

**Keywords:** spatial frame, deflection, double induction, asymptotics, Maple, analytical solution.

**Introduction.** Trusses offer a plethora of advantages over monolithic or sheet metal elements in building structures. Girder, arched and frame trusses are most commonly used as load-bearing structures. If the structure itself and the load can be decomposed into separate independent plane problems, the calculation of a spatial structure is considered in terms of the operation of the entire structure. Calculations of spatial trusses are commonly conducted in well-known numerical packages using the finite element method [11, 12, 15, 19, 20, 22, 23]. An alternative line of research into the operation of structures is developing analytical methods. Unlike the numerical ones, analytical solutions have such an advantage that they can be described by means of methods of analysis (e.g., for extremum points, for identifying asymptotes, jumps and inflection points). Nevertheless this applies only to those solutions that are applicable to a wide range of problems. If as a result of fairly complex work with analytical transformations in symbolic mathematics packages (Maple, Mathematica, Maxima, Derive, etc.), a solution is obtained that is applicable to a specific design and a specific load, the effort made is not justified. In this case, the numerical method yields all the basic values for evaluating a structure: deflection, distribution of forces in the rods and forces in critical rods. The more parameters are included in the calculation formula, the more efficient it actually is. The first fairly general formulas for calculating trusses, containing as parameters not only the dimensions and magnitude of the load, but also such an ordinal characteristic of regular systems as the number of panels were designed in the middle of the last century. The formula by Kachurin [6] is known, a number of formulas by Ignatiev [1, 2], algorithm for deriving analy-tical solutions for rod flat and spatial structures by Rybakov, including complex statically indeterminate [7, 8].

The relevance of designing a formula for dependence of the deflection of a spatial structure on the number of panels lies both in the need to have simple and reliable test solutions for the calculation and design of structures for evaluating the results obtained in numerical packages, and for comparing various options of schemes taking into consideration in the design process. Flat truss models do not allow analysis of the operation of a structure under the action of a load from its plane, e.g., a wind load. Therefore in this study a design scheme is proposed and considered which takes the work of the connections into account.

**1. Frame design.** The rectangular trussed frame has $m$ panels in height and $2n$ panels in the crossbar (Fig. 1, 2). The length of the panel in the crossbar is $a$, the height is $h$, and the width is $2b$. Let us place the structure in the coordinate system with the origin at the support $A$.

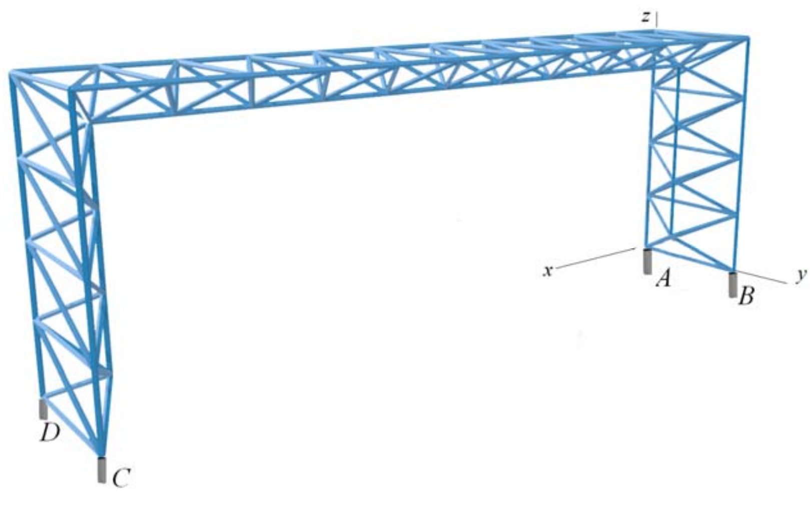


**Fig. 1.** Girder at $n = 5, m = 4$ : supports: $A$ is a spherical cylinder, $B$ is a cylindrical one, $C$ and $D$ are columns

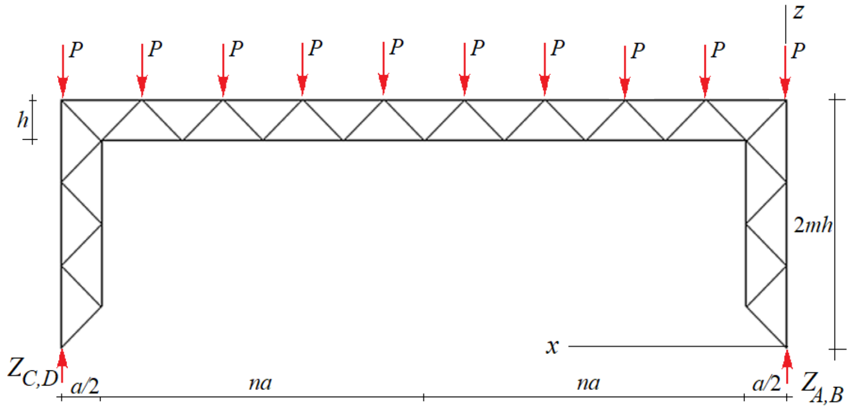


**Fig. 2.** Projection on the plane *x*-*z*. The girder under the action of an evenly distributed load at $n = 4, m = 3$: *P* is a load, *h, a* are the sizes, $Z_A$, $Z_B$, $Z_C$, $Z_D$ are vertical components of the support reactions

The elements of the girder are bar pyramids with a base $a \times 2b$ (Fig. 3) connected by additional longitudinal horizontal ties (bars of length *a*) along the tops.

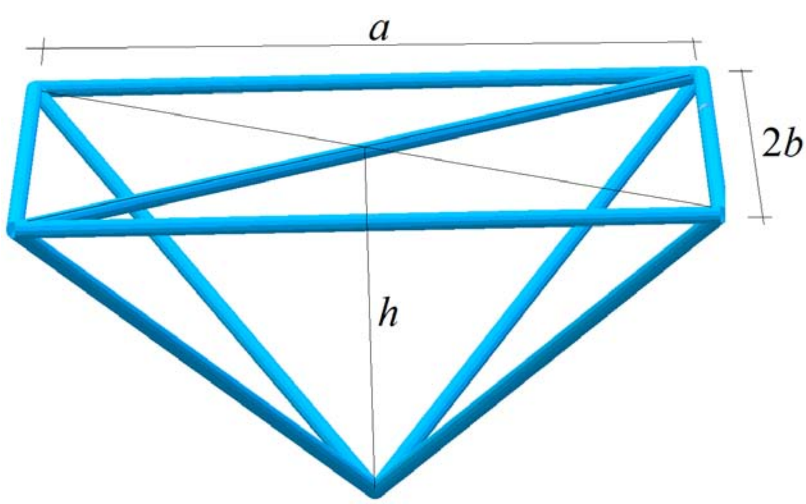


**Fig. 3.** Rod element of the crossbar element (panel): *a* is the length of the panel, 2*b* is the width, *h* is the height

Similar pyramids with the bases $2h \times 2b$ make up the symmetrically located lateral trusses with the height of 2*mh*.

The truss supports are a spherical hinge *A*, a cylindrical hinge *B*, and two posts at the angles *C* and *D*. There are in total $n_s = 18(n + m) + 3$ bars in the truss excluding the support ones.

The static indeterminacy caused by one extra support rod (support leg *D*) can be opened simply by replacing this rod with an external force found from the equilibrium condition of the entire system. With a load evenly distributed over 4(*n* + 1) nodes of the upper chord using the symmetry of the load and the structure, an effort is obtained. Therefore with this replacement taken into consideration, the construction is definable. All rods are assumed to be elastic and the hinges perfect.

The calculation of the deflection of the truss under the action of the load is performed based on the Maxwell-Mohr formula. The efforts in the rods are identified in symbolic form using the software [3] developed in the Maple computer mathematics system both for calculating plane [9, 10, 13, 14, 18, 24] and spatial [4, 5] statically definable trusses. The coordinates of the nodes of the truss are entered into the software. All the nodes (hinge joints of bars) are numbered (Fig. 4). E.g., the coordinates of the nodes of the reference trusses are as follows:

$$\begin{aligned}
&x_i = 0,\ y_i = 0,\ z_i = 2h(i-1),\ k = 2n+1,\\
&x_{i+m+k+1} = 2ak,\ y_{i+m+k+1} = 0, z_{i+m+k+1} = H - 2hi,\\
&x_{i+2m+k+1} = 0,\ \ y_{i+2m+k+1} = 2b, z_{i+2m+k+1} = 2h(i-1),\\
&x_{i+3m+2k+2} = 2ak, y_{i+3m+2k+2} = 2b,\ \ z_{i+3m+2k+2} = H - 2hi,\ i = 1,...,m.
\end{aligned} \tag{1}$$

$$\begin{aligned}
&m_1 = 2m + k + 1,\\
&x_{i+2m_1} = a,\ y_{i+2m_1} = b,\quad z_{i+2m_1} = h(2i-1),\\
&x_{i+5m+6n+4} = 0,\ \ y_{i+5m+6n+4} = b,\ z_{i+5m+6n+4} = h(2m-2i-a),\ \ i = 1,...,m-1.
\end{aligned}$$

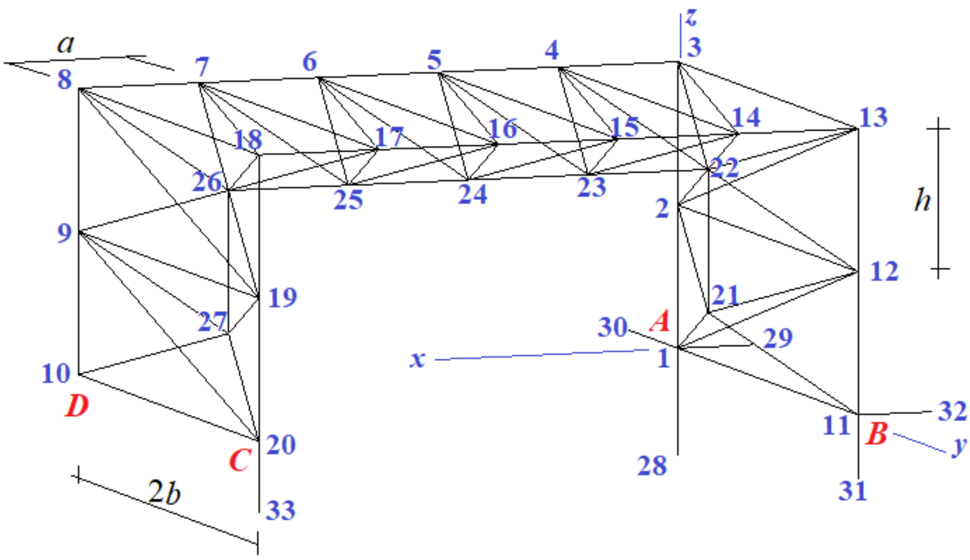


**Fig. 4.** Coordinate axes *x, y, z*, girder sizes *a, b, h,* numbering of the rods and nodes at $n = m = 2$

The structure of a truss is identified according to the order in which the members are connected. For that vectors are introduced with the numbers of the bars and components equaling those of the ends. The members of the truss outer contours, e.g., are defined according to the following vectors:

$$\overline{V}_i = [i, i+1],\ \ \overline{V}_{i+k+2m} = [i + m_1, i + m_1 + 1],\ i = 1,...,m_1 - 1.$$

**2. Solution. Vertical load.** For the efforts in the truss rods, a system of equations for all nodes is designed. For each node in the system, three equations are assigned in projection, respectively, on the *x, y, z* axis. The matrix *G* of the system includes the direction cosines of the forces calculated using the coordinates of a three-dimensional grid of nodes according to the data of the vectors $\overline{V}_i$, $i = 1,...,n_s + 6$ including six vectors modeling the supports. The load data is entered on the right side of the system. Under uniform loading of the nodes of the upper girder belt (Fig. 4), the vector of free members has the form $B_{3i} = P$, $B_{3(i+m_1)} = P$, $i = m+1,..,m+k+1$. The other components of this vector are zero. View components should contain projections of external forces applied to node *i* in projection onto the axis *x*, in the components $B_{3i-1}$ on the axis *y*. The efforts are determined using the solution. In the Maple system, the solution to a system of linear equations written in matrix form is most conveniently searched for by means of the inverse matrix method. In Maple it looks identical as when working with numbers. Here is the corresponding fragment of the program: $G1{:=}1/G$: $S{:=}G1$. Here *G1* is the inverse matrix, *S* is the vector of unknown efforts, *B* is the vector of the right sides of the system of equations. The dot in Maple denotes matrix multiplication or matrix-vector multiplication.

The deflection of the truss (the vertical displacement of the middle hinge in the lower girder chord) is identified by means of the Maxwell-Mohr formula

$$\Delta = \sum_{j=1}^{n_s} N_j \overline{N}_j l_j / (EF), \qquad (2)$$

where $N_j$ is the effort in the j-th bar of the truss from the applied load, $\overline{N}_j$ is the force in the same bar from a single vertical dimensionless force, $l_j$ is the length of the bar, *EF* is the stif-fness of the rods. Let us consider the case of a uniformly distributed load of vertical forces *P* distributed over the nodes of the upper belt at *h*=*a*. Sequentially calculating the trusses for *m* = 1 and *n* = 1, 2, 3, ... , we see that the form of the solution does not depend on the number of panels:

$$\Delta = P\frac{A_{n,m}a^3 + C_{n,m}c^3 + H_{n,m}b^3}{EFh^2}, \qquad (3)$$

where $c = \sqrt{a^2 + b^2 + h^2}$ and the coefficients for cubes of sizes form sequences whose common terms can be found using the Maple system operators. The coefficients $A_{n,m}$ at $a^3$ for *m*=1 have the following numerical sequence: 28, 198, 752,2050,4572, .... The rgf_findrecur operator identifies the recurrence equation that these numbers satisfy:

$$A_{n,1} = 5A_{n-1,1} - 10A_{n-2,1} + 10A_{n-3,1} - 5A_{n-4,1} + A_{n-5,1}. \qquad (4)$$

The solution of the equation is yielded by the rsolve operator:

$$A_{n,1} = 5n^4 + 10n^3 + 7n^2 + 4n + 2. \tag{5}$$

The set task of deriving the dependence of the deflection on the number of panels is a two-parameter one. At the second stage, it is essential to perform the same operations for $m = 2, 3, 4, \ldots$ for exactly as many times as necessary for the operator rgf_findrecur to yield a solution in the form of a homogeneous recurrent equation. We have the following results:

$$\begin{aligned} A_{n,2} &= 5n^4 + 10n^3 + 7n^2 + 4n + 4, \\ A_{n,3} &= 5n^4 + 10n^3 + 7n^2 + 6n + 6, \\ A_{n,4} &= 5n^4 + 10n^3 + 7n^2 + 8n + 8, \\ A_{n,5} &= 5n^4 + 10n^3 + 7n^2 + 12n + 10, \ldots \end{aligned} \tag{6}$$

The first three coefficients in these formulas do not change. The latter two generalize to arbitrary $m$ fairly simply without using the rgf_findrecur and rsolve operators:

$$A_{n,m} = 5n^4 + 10n^3 + 7n^2 + 2n(m+1) + 2m. \tag{7}$$

Similarly, but much simpler, the other coefficients are obtained. They do not depend on the number of panels $m$ along the truss height:

$$C_{n,m} = n(n+1), \;\; H_{n,m} = n. \tag{8}$$

In the case of a concentrated load, the deflection formula remains the same. The recurrent equations are simplified:

$$A_{n,1} = 4A_{n-1,1} - 6A_{n-2,1} + 4A_{n-3,1} - A_{n-4,1}\,. \tag{9}$$

The following solutions also take a more simple form:

$$\begin{gathered} A_{n,m} = 2n^3 + 3n^2 + 3n/2 + m/2 + 1/4, \\ C_{n,m} = (2n+1)/4, \;\; H_{n,m} = 1/4. \end{gathered} \tag{10}$$

**3. Lateral load.** The advantage of the spatial model of the structure compared to the widespread approximation of a truss by a set of flat trusses where the connections between them are conditionally not involved in the work is the possibility of calculating such models for an arbitrary load, e.g., for a rather rarely considered lateral load [16, 17, 25]. Let us look at the case of a load of horizontal forces $P$ uniformly distributed over the nodes of the upper chord of the truss. The matrix $G$ obtained for the vertical load remains the same. The inverse matrix $G1$ does not change either. The right-hand side composed of forces directed along the axis $y$ has the form $B_{3(i+m_1)-1} = P$, where $i = m+1, \ldots, m+k+1$. The force applied to the free node $D$ instead of the reaction of a nonexistent support is identified using the equation of equilibrium (moments) of the entire frame relative to the axis *x*: $Z_D = -Pm(n+1)h/b$. In this case, based on the solution of the system of equations for the equilibrium of the nodes, the expressions for the reactions of the supports are as follows: $Z_C = Z_B = Pm(n+1)h/b$, $Z_A = Z_D$.

The deflection formula (vertical displacement of the middle crossbar node in the lower chord) is also obtained by means of the induction method according to two parameters:

$$\Delta = P\frac{A_{n,m}a^3 + H_{n,m}b^3}{2EFhb}, \quad (11)$$

where

$$A_{n,m} = (n+1)((4m-2)n^2 + (8m-2)n + 2m - 1),$$
$$H_{n,m} = 2n^2 + (1+2m)n + 2m. \quad (12)$$

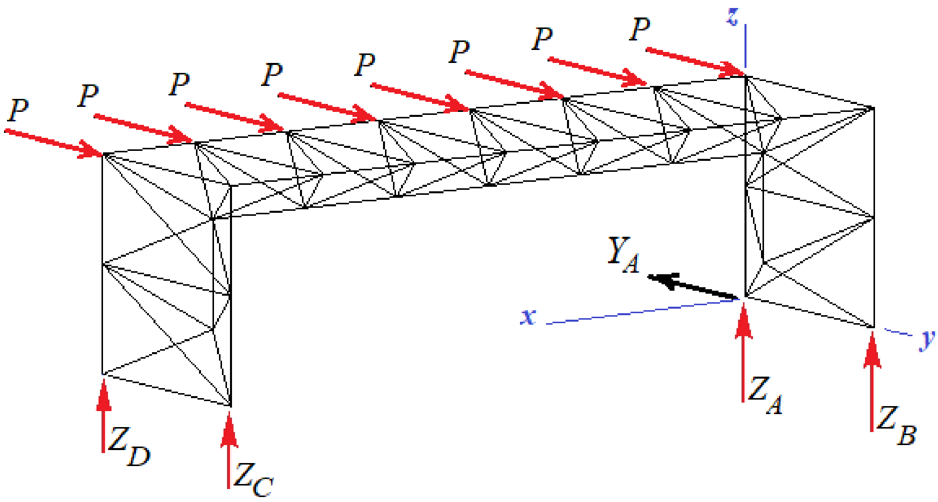


**Fig. 5.** Girder in the coordinates *x, y, z*. The lateral horizontal load *P* at $n = 3$, $m = 2$ : $Z_A$, $Z_B$, $Z_C$, $Z_D$ are vertical components of the support reactions, $Y_A$ is a horizontal component of the spherical hinge

**4. Analysis of the solution.** Let us consider the solution (3) with the coefficients (7), (8). Let us fix the value of the total load which does not depend on the number of panels $P_0 = P(2n+1)$ and design graphs of dependence on the number of panels of dimensionless deflection $\Delta' = \Delta EF/(P_0 L)$ where $L = 2an$ is the span length (Fig. 6). As the number of panels increases, so does the deflection. It can be established that this dependence tends to quadratic at $n \to \infty$ Indeed, $\lim_{n\to\infty} \Delta'/n^2 = 5/4$. A somewhat unexpected effect is also noticeable with an increase in the transverse dimension *b*. As b increases, the truss becomes wider (dimension along the axis *y*), the deflection grows. This can be partly accounted for by the fact that for such structures, the cross-links are lengthened, and in general the truss becomes less rigid not due to the longitudinal rods but due to the transverse ones.

The solution (3) with the coefficients (10) for a concentrated load has the same asymptotics and approximately the same curves.

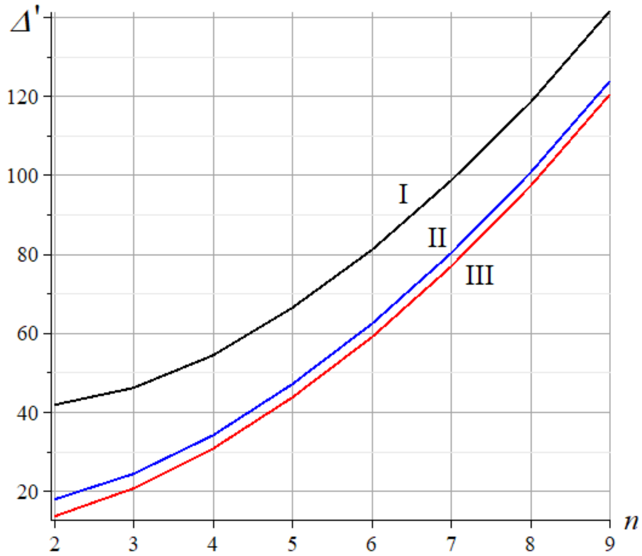


**Fig. 6.** Deflection under a load along the nodes of the upper belt, $m$ = 10, $a$ = 1 m, I — $b$ = 4m; II — $b$ = 2m; III — $b$ = 0.5m

The dependence of the dimensionless deflection on the panel size under the action of a lateral horizontal loadcalculated according to the solution (11, 12) has a distinct horizontal asymptote $\tilde{\Delta}$ limiting the solution from the bottom (Fig. 7).

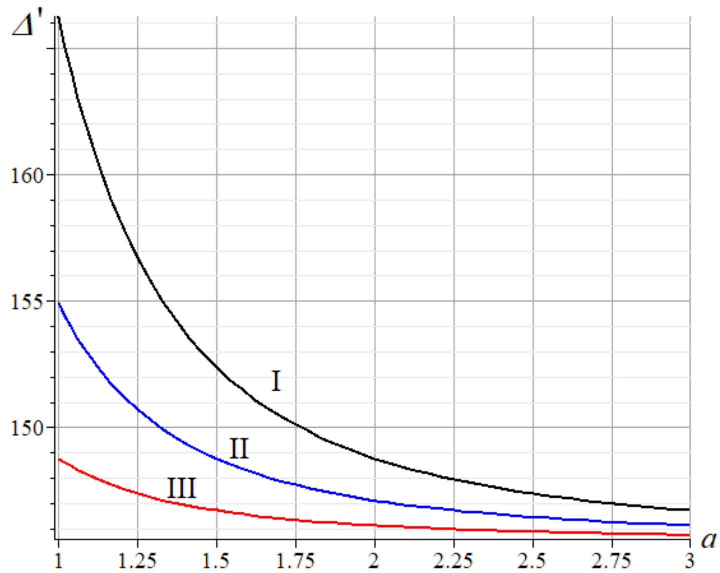


**Fig. 7.** Dependence of the deflection on the size of the panel under the action of the lateral load, $n = 10$, $m = 3$, I — $b$= 4 m; II — $b$ = 3 m, III — $b$=2 m

The value of the ultimate deflection is obtained in the Maple system using the limit (Del, a = infinity) operation and is as follows:

$$\tilde{\Delta} = \frac{(5n^3 + 5n^2 + 2n + 2\sqrt{2}\,n + 2m)(n+1)}{2n(2n+1)} \;. \quad (13)$$

Let us note that this value depends only on the number of panels and not on the linear dimensions of the structure. Obviously, as $n$ or $m$ increase, the value grows indefinitely.

**Conclusions.** The suggested scheme of a spatial frame with two parameters that regulate the number of panels in the girder and uprights makes it possible for the inductive method to be applied in order to obtain the basic formulas for assessing structural deformations. It is convenient to use these estimates for optimization problems [21] as well as test estimates for evaluating numerical solutions [22]. The suggested truss scheme is new, the analytical solutions for its deflection have been obtained for the first time.

In the process of deriving formulas in a system of symbolic transformations which works much slower than packages employing numerical methods, a difficulty of a purely technical nature had to be overcome. As the number of panels in the girder increases, the counting time increased sharply and since the induction in this task was double, either a processor with good characteristics or a significant counting time was needed. If at first the induction is performed using one parameter in $N$ steps and time $t$ is spent on each step, the induction on another parameter in $M$ steps already requires time NM$t$. The solution to this problem was somewhat simplified in cases where the individual coefficients did not depend on the number $m$ of panels in height - the second-order induction parameter. The formulas were verified in two ways: numerically or by changing the order of the induction parameters.

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