Lower estimate of the fundamental frequency of natural oscillations of a truss with an arbitrary number of panels

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ABSTRACT

Introduction: the paper deals with oscillations of a statically definable plane, truss with a double lattice of racks and descending braces with massive loads in the nodes of the lower chord. The weight of the truss rods is not taken into account. It is assumed that the freights are moved only vertically. The fundamental frequency of natural oscillations is estimated from the Dunkerley formula by the values of partial frequencies.

Materials and methods: an analytical estimate is obtained by generalizing formulas obtained from a series of estimates for trusses with a consistently increasing number of panels. The stiffness of the truss was determined using the Mohr's integral. The double lattice of the truss does not allow using the cross-section method; therefore, the forces in the rods were calculated (or estimated) in an analytical form using the method of cutting nodes with the compilation of a system of equilibrium equations simultaneously for all rods and three support reactions. The matrix of equilibrium equations was compiled in a software program written in the language of the Maple computer mathematics system based on the coordinates of the nodes and the values of the direction cosines of the forces. For a sequence of coefficients of the desired formula, linear homogeneous recurrent equations were found and solved by means of special operators of the Maple system.

Results: the resulting formula estimating the relationship between the fundamental frequency and the panels number has the form of a sixth degree polynomial with coefficients depending on the parity of the number of panels. The analytical result is compared with the smallest frequency obtained numerically from the solution of the problem of oscillation of the cargo system. It is shown that the main frequency, depending on the truss height, has an extremum.

Conclusions: the method of generalizing particular solutions using the Maple system operators allowed authors to obtain and analyze a formula for a lower estimate of the fundamental frequency of oscillation of a truss model with an arbitrary number of panels. The resulting estimate can be used as a test for numerically obtained solutions. The formula is especially efficient for systems with a large number of panels; as numerical methods for their calculation are time-consuming require considerable time and have a tendency for accumulating rounding errors.

KEYWORDS: truss, natural oscillations, fundamental frequency, induction, lower frequency estimate, Maple

Нижняя оценка основной частоты собственных колебаний фермы с произвольным числом панелей

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АННОТАЦИЯ

Введение: рассмотрены колебания статически определимой плоской фермы с двойной решеткой из стоек и нисходящих раскосов с массивными грузами в узлах нижнего пояса. Масса стержней фермы не учитывается. Предполагается, что грузы перемещаются только вертикально. Основная частота собственных колебаний оценивается снизу по формуле Донкерлея по значениям парциальных частот.

Материалы и методы: аналитическая оценка получена обобщением формул, выведенных по ряду оценок для ферм с последовательно увеличивающимся числом панелей. Жесткость фермы определена с помощью интеграла Мора. Двойная решетка фермы не позволяет применять метод сечений, поэтому усилия в стержнях находятся в аналитической форме методом вырезания узлов с составлением системы уравнений равновесия одновременно для стержней и трех реакций опор. Матрица уравнений равновесия составлена в программе, написанной на языке системы компьютерной математики Maple по данным о координатах узлов и значениям направляющих косинусов усилий. Для последовательности коэффициентов искомой формулы с помощью специальных операторов системы Maple найдены и решены линейные однородные рекуррентные уравнения.

Результаты: полученная формула зависимости основной частоты от числа панелей имеет вид полинома шестой степени с коэффициентами, зависящими от четности числа панелей. Аналитический результат сравнивается с наименьшей частотой, полученной численно из решения задачи о колебании системы грузов. Показано, что основная частота в зависимости от высоты фермы имеет экстремум.

Выводы: метод обобщения частных решений с применением операторов системы Maple позволил получить и проанализировать формулу для нижней оценки основной частоты колебания модели фермы с произвольным числом панелей. Данная оценка может быть использована как тестовая для решений, полученных численно. Формула особенно эффективна для систем с большим числом панелей, для расчета которых численные методы требуют значительное время и имеют свойство накопления ошибок округления.

КЛЮЧЕВЫЕ СЛОВА: ферма, собственные колебания, основная частота, индукция, нижняя оценка частоты, Maple

INTRODUCTION

Frequency characteristics of building structures are required both for designing structures and for evaluating various modes of their operation. Traditionally, structural engineers solve the problem of estimating natural frequencies by numerical methods in standard packages, using mainly the finite element method [1–6]. Analytical calculations of natural frequencies resulting in a compact formula are rather rare [7–10]. In most solutions, reduced in analytical form only to the coefficients of the characteristic equations, regularity property of a calculated truss is applied. The periodicity of the truss structure [11–14] allows applying the induction method in order to obtain the dependence of the solution on the panels number. Nonlinear oscillations of trusses were studied in [15]. In [16–18], oscillations of flat elastic trusses with an orthogonal structure were studied with allowance for internal friction under the Voigt model by the method of "gluing" solutions. The dynamics of a truss with a triangular lattice and parallel belts was studied in [19]. Oscillations of truss nodes and wave processes related to them are considered in [20]. Analytical expressions for the frequencies of natural oscillations of loads concentrated in the nodes of flat trusses were obtained in the Maple system by the induction method in [21]. Papers [22–24] give some experimental results in comparison with theoretical calculations of truss oscillation frequencies.

MATERIALS AND METHODS

As an object of study, the authors have chosen a flat truss with a double lattice of descending bracings and struts (Fig. 1). To derive the formula for the dependence of the fundamental frequency of natural oscillations on the panels number, the authors have applied the software program for calculating the forces in the rods of a statically definable truss written in the language of symbolic mathematics Maple [25].

The truss has $n_0 = 8(n + 1)$ rods, including $2n$ belt rods of length a, $2(n + 1)$ braces and $2n - 1$ racks. Modeling the inertial properties of the structure, the same concentrated masses m were put at the nodes of the lower belt while the mass of the rods was neglected. For practical purpose. the main, lowest natural frequency of oscillation of the truss is of great interest. One can predict (or guesstimate) (this could be further verified in the research paper) that a nonlinear frequency equation of $2n - 1$ order in the general case will not allow one to find this frequency in analytical form [7–10]. For the lower estimate of the desired frequency $\tilde{\omega}$, the well-known and sufficiently accurate lower Dunkerley estimate was applied. The formula has the form

$$\tilde{\omega}^2 = 1 \bigg/ \sum_{k=1}^{2n-1} 1/\omega_k^2, \qquad (1)$$

where ω_k is the frequency of oscillation of the load m_k in the node $k + 1$ of the lower belt, in the absence of all
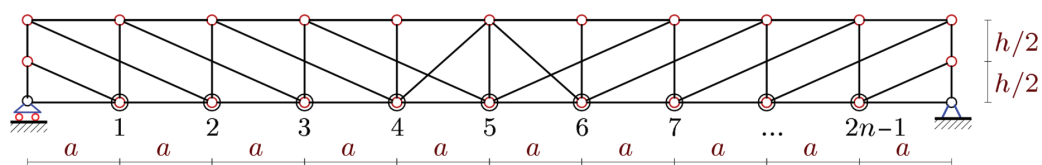

Fig. 1. Truss with weights in the nodes of the lower belt, $n = 5$

other masses (the numbering of the nodes goes from left to right, starting from the mobile support).

The differential equation of mass oscillations $m_k = m$ is

$$m\ddot{y}_k + d_{k,n}y_k = 0,$$

where y_k is the vertical mass displacement, $\ddot{y}_k$ is the acceleration vector, $d_{k,n}$ is the stiffness coefficient (k is the mass number, n is the number of panels). Taking the standard form $y_k = A_k \sin(\omega_k t + \varphi_0)$ of deflection in such cases, the formula for estimating the partial frequency was obtained

$$\omega_k = \sqrt{d_{k,n}/m}.$$

Stiffness coefficient can be calculated by means of compliance coefficient, using the Maxwell – Mohr formula. We get:

$$\delta_{k,n} = 1/d_{k,n} = \sum_{v=1}^{n_0-3} \frac{S_v^{(k)2}l_v}{EF}.$$

Here $S_v^{(k)}$ is the force in the rod with the length l_v from the unit vertical force applied to the node k, where the mass is located, while EF is the stiffness of the rod. The sum is compiled for all rods of the truss, apart from three, modeling supports. Thus, we obtain an estimate

$$\tilde{\omega} = \sqrt{1 / \sum_{k=1}^{2n-1} \delta_{k,n} m} = \sqrt{1/(\Lambda_n m)}, \qquad (2)$$

where indicated $\Lambda_n = \sum_{k=1}^{2n-1} \delta_{k,n}$. If we set the task of deriving the analytical dependence of the oscillation frequency on the number of panels, then the loadings included in (1) should also be defined in symbolic form. The algorithm for calculating the forces in the truss in the computer system of symbol mathematics Maple is well developed and tested when solving problems of deflection [25]. Using the program [25], we obtain a series of solving problems on the fundamental oscillation frequency of a truss with a successively increasing number of panels. The solution for the coefficient Λ_n is

$$\Lambda_n = \frac{C_{1,n}a^3 + C_{2,n}c^3 + C_{3,n}h^3 + C_{4,n}d^3}{2n^2h^2EF}, \qquad (3)$$

where $c = \sqrt{a^2 + h^2}$, $d = \sqrt{4a^2 + h^2}$ are the lengths of braces. The dependence of the coefficients on the panels number has been received by induction method. We have the first four expressions:

$$\Lambda_1 = \frac{a^3 + c^3 + 2h^3}{2h^2EF},$$

$$\Lambda_2 = \frac{2(64a^3 + 8c^3 + 15h^3 + 7d^3)}{8h^2EF},$$

$$\Lambda_3 = \frac{855a^3 + 43c^3 + 145h^3 + 91d^3}{18h^2EF},$$

$$\Lambda_4 = \frac{8(504a^3 + 16c^3 + 55h^3 + 39d^3)}{32h^2EF}, \dots$$

In order to generalize these solutions and to obtain the desired frequency, dependence on an arbitrary number n, the four solutions are not considered sufficient. The required length of the solutions sequence is determined by the capabilities of the **rgf_findrecur** operator of the **genfunc** package from the Maple system, which returns a recurrent equation to which the sequence members obey. In this case, the length turned out to be rather large compared with similar solutions previously obtained for deflection problems [25, 26]. To determine the coefficient $C_{1,n}$ at a^3, it is necessary to analyze 22 solutions, that could be time- and energy-consuming, taking into account the very low speed of symbolic transformations.. As a result of processing the sequence 1, 128, 855, 4032, 13 645, ..., 61 696 509, 81 474 624, a linear homogeneous recurrent equation of the eleventh order was obtained:

$$C_{1,n} = 2C_{1,n-1} + C_{1,n-2} - 11C_{1,n-3} + 6C_{1,n-4} +$$
$$+ 14C_{1,n-5} - 14C_{1,n-6} - 6C_{1,n-7} +$$
$$+ 11C_{1,n-8} - C_{1,n-9} - 3C_{1,n-10} + C_{1,n-11}.$$

The solution of this equation with the initial data $C_{1,1} = 1$, $C_{1,2} = 128$, $C_{1,3} = 855$, ... is given by the **rsolve** operator:

$$C_{1,n} = n(64n^5 + 310n^3 + (375 - 15(-1)^n)n^2 +$$
$$+ (225(-1)^n - 329)n + 60((-1)^n - 1))/90. \qquad (4)$$

Similarly, to estimate the coefficient $C_{2,n}$, a sequence of length 16 was required. The equation obtained from this sequence has the form

$$C_{2,n} = 4C_{2,n-2} - 6C_{2,n-4} + 4C_{2,n-6} - C_{2,n-8}.$$

The solution of the equation is the common term of the corresponding sequence:

$$C_{2,n} = n(((-1)^n + 11)n^2 + 2((-1)^n - 1))/6. \qquad (5)$$

For the coefficient $C_{3,n}$, we get the ninth order equation

$$C_{3,n} = C_{3,n-1} + 4C_{3,n-4} - 4C_{3,n-3} - 6C_{3,n-4} +$$
$$+ 6C_{3,n-5} + 4C_{3,n-6} - 4C_{3,n-7} - C_{3,n-8} + C_{3,n-9}.$$

The solution of the equation has the form of a fourth degree polynomial.

$$C_{3,n} = n(18n^3 + (13 - (-1)^n)n^2 +$$
$$+ 3((-1)^n - 3)n + 2(1 - (-1)^n))/12. \qquad (6)$$

For the coefficient $C_{4,n}$, the equation turns out to be the same as for $C_{3,n}$, and the solution differs from $C_{3,n}$ only in one term:

$$C_{4,n} = n\left(18n^3 - \left(11 + (-1)^n\right)n^2 + \right.$$
$$\left. + 3\left((-1)^n - 3\right)n + 2\left(1 - (-1)^n\right)\right)\Big/12. \quad (7)$$

Thus, dependence (2)–(3) with coefficients (4–7) determining the lower estimate of the fundamental oscillation frequency for any number of truss panels, gives the solution to the problem.

RESULTS

Consider, for example, a conventional truss made of steel rods with an elastic modulus of $E = 2 \cdot 10^5$ MPa and cross section $F = 1$ sm^2. Assumed panel length $a = 1$ m, the calculation was made for the three truss options with the number of panels $n = 10$, $n = 11$ and $n = 12$ in half span. In the plots (Fig. 2) of the dependence of the fundamental frequency (rad/s) on height h (in meters) extremes are observed. This feature of the solution can be applied in truss design while finding the most optimal value of the natural frequency of the structure.

The degree of approximation of the obtained estimate to the exact solution can be calculated only by solving the problem of oscillation of a $2n - 1$ load system. In this case, the oscillations are described by a system of differential equations of order $2n - 1$.

$$m\ddot{\overline{Y}} + [D_n]\overline{Y} = 0, \quad (8)$$

where $[D_n]$ is the stiffness matrix for the truss with n panels. Taking into account the usual replacement of the $\overline{Y} = \overline{A}\sin(\omega t + \varphi_0)$, equivalent to the substitution $\ddot{\overline{Y}} = -\omega^2 \overline{Y}$ in such problems, we multiply (8) on the left

by the compliance matrix $[B_n]$, the inverse of the stiffness matrix $[D_n]$. We obtain a homogeneous equation $m\omega^2 [B_n]\overline{Y} = \overline{Y}$ for the displacement vector that narrows the problem to determining the eigenvalues of the matrix $[B_n]$

$$[B_n]\overline{Y} = \lambda \overline{Y}.$$

The natural frequency is equal to $\omega_k = 1/\sqrt{\lambda_k m}$. We define the compliance matrix $[B_n]$ by the Maxwell-Mohr's formula $b_{i,j} = \sum_{\nu=1}^{n_s-3} S_\nu^{(i)} S_\nu^{(j)} l_\nu \Big/ (EF)$, where $S_\nu^{(i)}$ is the force in the rod ν from the action of a single vertical force at node i, $S_\nu^{(j)}$ is the force in the rod ν from the action of a single vertical force at node j. Figure 3 shows a comparison of the fundamental frequency received numerically (curve 1) and the found analytical estimate for Dunkerley (2). Calculations are performed for the same data as for the graphs in Fig. 2.

The spectrum of a truss with a different number of panels ($n = 1, 2, ..., 6$) according to the numerical solution data is displayed by the curves in Fig. 4. The frequencies in the spectrum grow non-monotonously, which is especially evident for trusses with a large number of panels. There is a noticeable sharp jump at the end of the spectrum.

It is also easy to obtain an analytical solution on the base of a numerical one applying the same scheme in the Maple system. If obtaining a solution in the form of formulas depending on the panels number is not required, then the first few eigen frequencies of the system with a small number of panels can be written out explicitly. So, with $n = 2$, the compliance matrix is
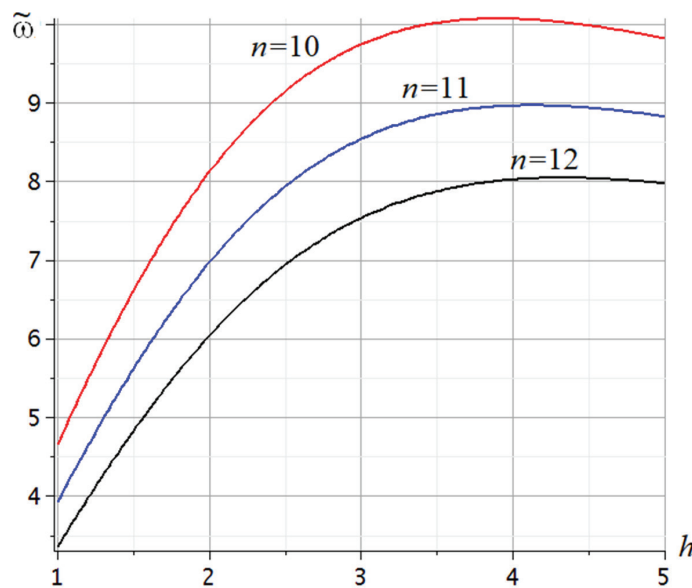

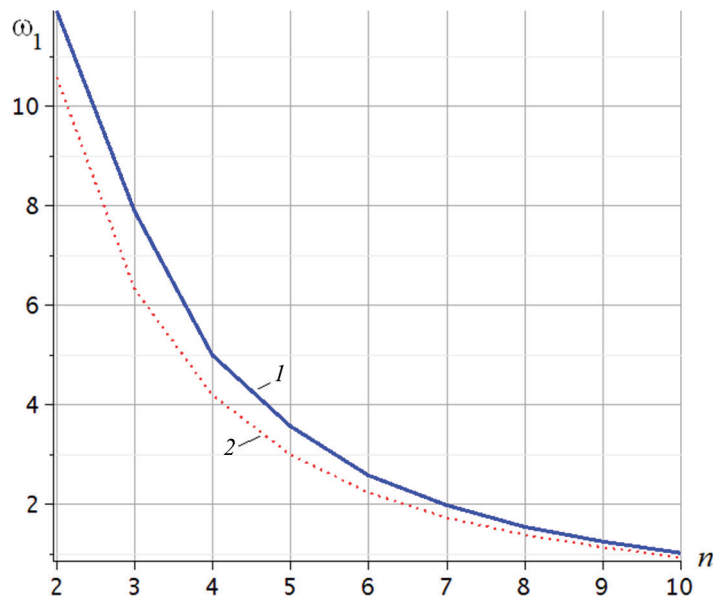**Fig. 2.** Dependence of the main frequency on the truss height

Fig. 3. The fundamental frequency, $a = 3$ m, $h = 1$ m. Comparison of solutions: *1* — numerical; *2* — analytical evaluation
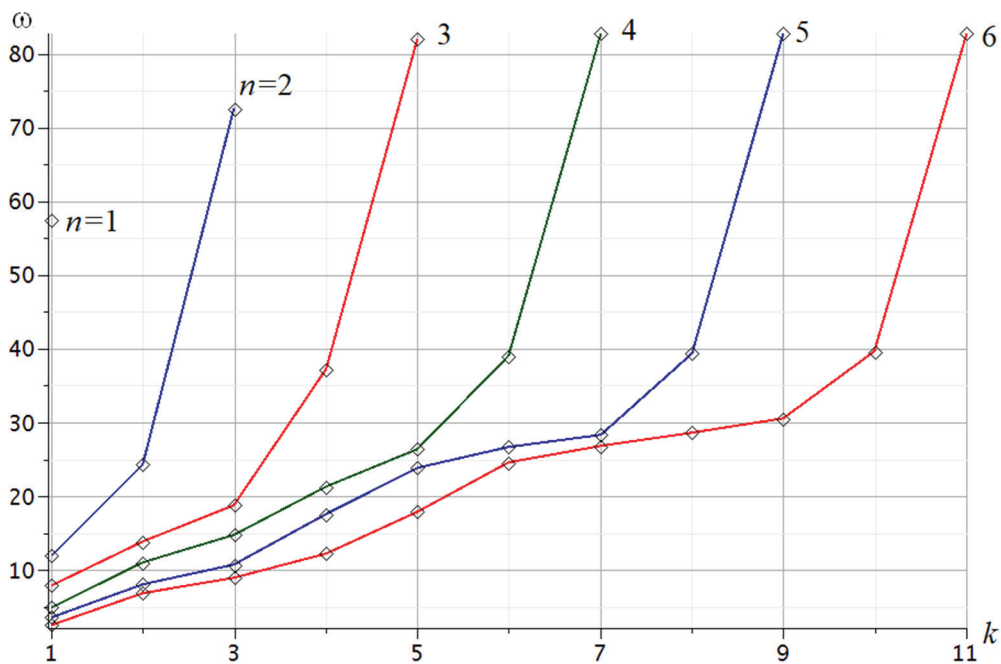

Fig. 4. Spectra of trusses with different number of panels (rad/s), k — number of frequency in the spectrum

$$B_2 = \frac{1}{8h^2 EF}\begin{bmatrix} 48a^3 + 8c^3 + 5d^3 + 13h^3 & 4(8a^3 + d^3 + h^3) \\ 4(8a^3 + d^3 + h^3) & 4(8a^3 + d^3 + h^3) \\ 24a^3 + 3d^3 + 11h^3 & 4(8a^3 + d^3 + h^3) \end{bmatrix}$$

$$\begin{bmatrix} 24a^3 + 3d^3 + 11h^3 \\ 4(8a^3 + d^3 + h^3) \\ 48a^3 + 8c^3 + 5d^3 + 13h^3 \end{bmatrix}.$$

One of the roots of the frequency equation is

$$\lambda_1 = \left(12a^3 + 4c^3 + d^3 + h^3\right) / \left(4EFh^2\right).$$

The two other eigenvalues corresponding to the highest and lowest frequency are the roots of the quadratic equation:

$$\lambda^2 - 2h^4 \left(26a^3 + 2c^3 + 3d^3 + 7h^2\right)\lambda EF +$$
$$+ 8h^8 \left(8a^3 + d^3 + h^3\right)\left(a^3 + c^3 + 2h^3\right)E^2 F^2 = 0.$$

CONCLUSIONS

Considered truss scheme is not the easiest for a structural engineer. The double lattice does not permit the use of simple methods of calculating the forces, such as the method of sections or the method of successive

knots cutting. Therefore, when calculating loading, it is required to compose and solve a system of equilibrium equations for all nodes at once; and this could lead to some technical issues, in particular when calculating a truss with a large number of panels. When using symbolic transformations, the problem of count time is added to ordinary problems. This is due to the fact that symbolic transformations are performed much slower than symbolic ones. If this problem did not exist, then the formula of frequency dependence on the panels number of panels would not be required either. It would be sufficient just to set the required number of panels in the software program functioning in the Maple system and the program would give the desired dependence of the oscillation frequency on the size of the truss and the cargo mass. However, practice shows that on computers of average power (the author worked on a computer with an i7 processor), the result can actually be obtained for a truss with 20–25 panels. The calculation of trusses with a large number of panels does not require hours, but days of work, that is not acceptable for an engineer, especially if you need to choose the number of panels, i.e. perform several calculations. Inductive method allows eliminating all the above-mentioned problems. Sequential counting for trusses with the number of panels *n* from 1 to 10 makes it possible to identify a pattern in the coefficients of the desired formula and to obtain a universal dependence in this formulation. Another feature of the obtained solution is the successful application of the Dunkerley estimate for fundamental frequency. This is due to the fact that the authors managed to find the exact and rather compact value of the final sum determining the frequency. A specific numerical example showed that the lower bound had proven to be quite accurate. The applied algorithm and the resulting formula can be put in practice by structural engineers for structural design.

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