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ANALYTICAL CALCULATION OF DEFORMATIONS AND KINEMATIC ANALYSIS OF A FLAT TRUSS WITH AN ARBITRARY NUMBER OF PANELS

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Statement of the problem. Analytical dependences of the deflection and displacement of the support of a flat lattice truss on the number of panels are being sought. The truss has a double lattice, a rectilinear lower belt and an upper belt raised in the middle part.
Results. For two types of loading, according to the Maxwell-Mohr formula, analytical dependences of the deflections of the structure on the load, dimensions and number of panels are obtained. To generalize a series of particular solutions for trusses with different numbers of panels for an arbitrary case, the induction method and the analytical capabilities of the Maple computer mathematics system were used. For some solutions, asymptotic approximations are obtained. The distribution of forces in the rods of the structure is shown.
Conclusions. The obtained formulas can be used in optimization problems and as test ones for evaluating approximate numerical solutions. Cases of geometric variability of the truss with the number of panels being a multiple of three are revealed. An algorithm for identifying the corresponding distribution of possible velocities of the joints is presented.

Keywords: flat truss, deflection, support displacement, induction, asymptotics, Maple, analytical solution, geometric variability.

Introduction. One of the most common types of trusses are lattice trusses. The paper swets forth a scheme of this exact type and provides a fairly general formula for the dependence of its deflection, the displacement of the movable support and the first frequency of natural oscillations on the dimensions of the truss and the number of panels (Fig. 1).

The truss belongs to regular constructions [2, 4]. For the analytical calculation of the stress-strain of such structures with an arbitrary number of periodicity cells (panels), the inductive approach [1, 20] is used. Theoretical questions regarding the existence and analysis of regular

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planar and spatial statically determined trusses were previously addressed by S. D. Guest, R. G. Hutchinson and N. A. Fleck [11—13].

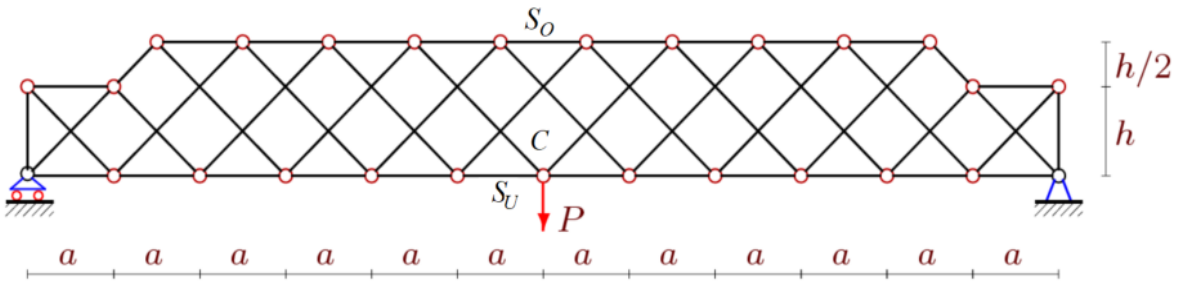

Fig. 1. Girder at $n = 5$ under a concentrated load

A special feature of the suggested scheme is that due to the complex lattice, such simple methods as the section method (there are no Ritter sections here) and the method of sequential cutting of nodes are not applicable to it, as there is no initial node (hinge) connected to two rods in the farm with unknown effort. A general system of equilibrium equations for all nodes is compiled and solved for the calculation including the reactions of the supports as well. Therefore according to the condition of the task, the number of panels in the design is arbitrary leading to a solution to large-order systems and the need to involve a computer mathematics system. If we limit ourselves only to the definition of efforts, any system is appropriate (*Maple, Mathematica, Maxima, Derive*, etc.). However, in order to derive general formulas by induction, the system must have specially designed operators. The most convenient system turned out to be *Maple* [10], in the language of which there is a program previously used to investigate the deflection of flat [7, 8, 14, 18, 19] and three-dimensional girders [3, 16, 17]. In combination with the operators of this system, the method of induction was also used for calculating the natural frequencies of flat statically determinate trusses. For analytical calculations of rod systems, including statically indeterminate ones, L. S. Rybakov's algorithm is also used [5, 6].

1. Truss design and force calculation. A flat truss has $2n + 2$ panels of length a along the bottom chord, height $3h / 2$ in the middle and h above the supports. The left support is pivotally movable, the right one is fixed. There are $n_s = 8n + 14$ bars in the truss, including three bars modeling supports. The lower rectilinear chord contains $2n + 3$ hinges, the upper one — $4n + 4$. The farm is statically determinate. In order to identify the forces in the rods, the program in the *Maple* language [1] is used enabling us to find the forces in the rods in an analytical form. Rods and nodes are numbered. E.g., the coordinates of the hinges of the lower chord look in the following way (the origin of coordinates in the hinge of the movable support):

$$x_i = (i-1)a,\ y_i = 0,\ i = 1,\ldots,2n+3.$$

The coordinates of the upper belt are

$$x_{2n+4} = 0, x_{2n+5} = a,\ x_{4n+6} = x_{2n+2},\ x_{4n+7} = x_{2n+3},$$
$$y_{2n+4} = y_{2n+5} = y_{4n+6} = y_{4n+7} = h,$$
$$x_{i+2n+5} = (i+1/2)a,\ y_{i+2n+5} = 3h/2,\ i = 1,...,2n.$$

The structure of the girder is specified by ordered lists corresponding to the rods. For the elements of the lower belt, we have lists:

$$\overline{V}_i = [i, i+1], \quad i = 1,...,2n+1.$$

The upper belt is encoded in the following way:

$$\overline{V}_{i+2n+2} = [i+2n+3, i+2n+4], \quad i = 1,...,2n+3.$$

Similarly, bars of the lattice are encoded in cycles of length $2n + 1$.

A system of equilibrium equations for all the nodes is compiled in projections onto the coordinate axes $\mathbf{G}\overline{S} = \overline{\mathbf{B}}$. For each node in the system, two equations are assigned in the projection on the x and y axes respectively. The elements of the matrix G of the system are the direction cosines of the forces calculated from the coordinates of the nodes and in compliance with the data of the lists $\overline{V}_i,\ i = 1,...,n_s$

The right part of the system is a load vector $\overline{\mathbf{B}}$. Vertical forces are placed in the even elements of this vector, and horizontal forces in the odd ones. For uniform loading of the nodes of the lower belt (Fig. 4) we have:

$$B_{2i} = P,\ i = 2,..,2n+2\ .$$

The remaining components of this vector equal zero. Efforts are identified based on the solution of the system $\mathbf{G}\overline{S} = \overline{\mathbf{B}}$. In the *Maple* system the solution of a system of linear equations compiled in matrix form is identified by means of the inverse matrix method. The inverse matrix is identified easily in *Maple*. The corresponding fragment of the software looks in the following way: *G_*1:=1/*G*: *S*:=*G_*1. *B*. Here *S* is the vector of unknown forces; *B* is the vector of the right parts of the system of equations; *G_1* is the inverse matrix. The multiplication of a matrix by a vector in the *Maple* language is denoted by a dot.

2. Random geometric variability. The first calculations of the truss at different values that are common for systems are called to zero if the number of panels is a multiple of three. In order to confirm this, a scheme of possible speeds of the nodes must be designed. To this end,

it is convenient to regulate the rotation speed of the middle hinge of the two-link link based on the speeds and coordinates of the hinge and the speed of the rods. We have a double link *i, j, k* (Fig. 2). Components: $v_{x,i}$, $v_{y,i}$ speed joints *i* are part of the line connection system:

$$v_{x,j} = v_{x,i} - (y_j - y_i)\omega_{ij},$$
$$v_{y,j} = v_{y,i} + (x_j - x_i)\omega_{ij},$$
$$v_{x,k} = v_{x,i} - (y_k - y_i)\omega_{ik},$$
$$v_{y,k} = v_{y,i} + (x_k - x_i)\omega_{ik}.$$

where ω_{ij}, ω_{ik} are angular speeds of the links.

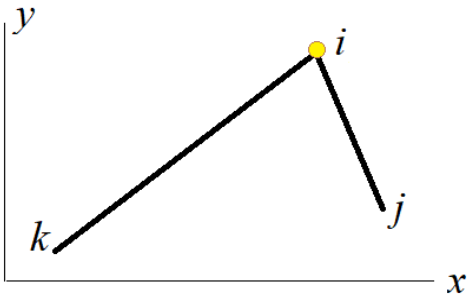

Fig. 2. A double link for identifying speeds

It is convenient to represent the calculation schemes in the form of a graph meaning that the speed of points with the number *i* is calculated based on the data of the speeds of points *k* and *j*. In considering the truss at *n* = 3 (Fig. 3), we assume that the speed of hinge 5 is equally loaded, and the speed of rotation of hinges 16 and 13 is provided that will be denoted as *v*. These vectors are directed perpendicularly to the hinges 5–16 and 5–13, respectively. Next, we sequentially design graphs $\{5,16\}\to 8$, $\{9,8\}\to 19$, $\{9,19\}\to 18$, $\{8,18\}\to 7$, $\{16,18\}\to 17$, $\{7,17\}\to 6$, $\{7,16\}\to 15$, $\{6,15\}\to 14$, $\{5,14\}\to 13$, $\{5,15\}\to 4$, $\{4,14\}\to 3$, $\{3,13\}\to 2$, $\{4,13\}\to 12$, $\{3,12\}\to 11$, $\{2,11\}\to 10$.

As a result, we get

$$v_2 = v_4 = v_6 = v_8 = 2va/c, \quad v_3 = v_7 = 4va/c,$$
$$v_{10} = v_{19} = u = 2vh/c, \; v_{11} = v_{18} = 2v,$$
$$v_{12} = v_{13} = v_{16} = v_{17} = v, \; v_{15} = v_{14} = u' = v\sqrt{9a^2 + h^2}/c,$$

where $c = \sqrt{a^2 + h^2}$.

Hence a kinematically consistent field of variability rates of a variable system is identified using a rare specific system with a characteristic arrangement. It should be noted that sufficient geometric characteristics of variability do not work here. In this case, the conditions for

switching on a closed circuit are not adapted either [15]. The suggested algorithm for identifying the velocity field has a feature associated with the choice of the major points of the graph chain. Here the hinge 5 is chosen with a denoted zero speed. The disadvantage of the algorithm is that the starting point is in chosen experimentally, simply by randomly going through the options. E.g, if we start from the hinge 10 moving from left to right, the sequence $\{1,10\} \to 11$, $\{1,10\} \to 2$ of the double links iwth two known speeds (previously identified) is disrupted after the graph $\{2,11\} \to 3$.

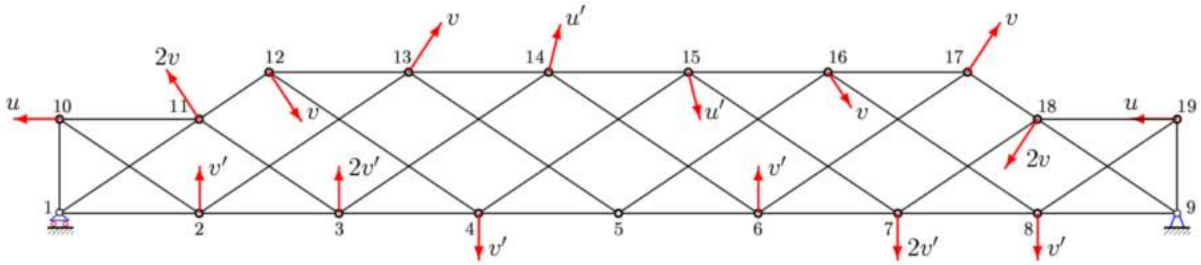

Fig. 3. Variable truss speed distribution scheme, $n = 3$

The same scheme of possible velocities can be obtained for other cases of n divisible by three. We exclude these cases from the search introducing a new variable

$$n = (6k - (-1)^k - 3)/4 \qquad (1)$$

for obtaining the dependence of the solutions on the number of panels.

The kinematics of plastic fracture of three-dimensional trusses was discussed in [9].

2. Calculation of efforts. The solution of the system of linear equations in the *Maple* system yields analytical expressions for all the efforts. Additionally, the *Maple* operators allow us to obtain a visual picture of the distribution of forces in the rods in the numerical mode of this program. With $a = 3$ m, $h = 2$ m in the case of a distributed load (Fig. 4), we design a force diagram (Fig. 5).

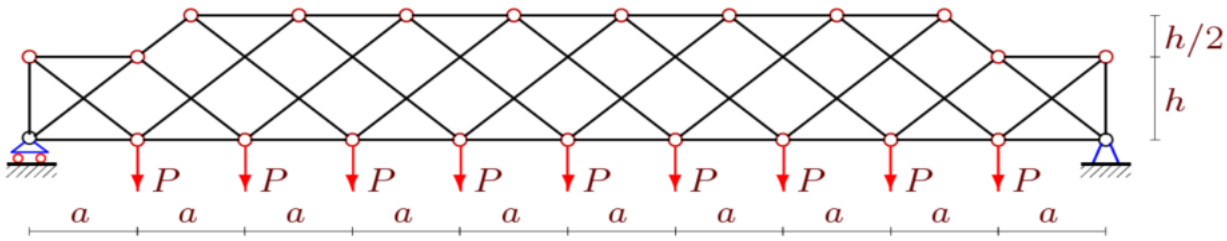

Fig. 4. Truss under the action of a uniformly distributed load along the lower belt with $n = 4$

Stretched rods are highlighted in red, compressed rods in blue. The value of efforts is related to the value of P.

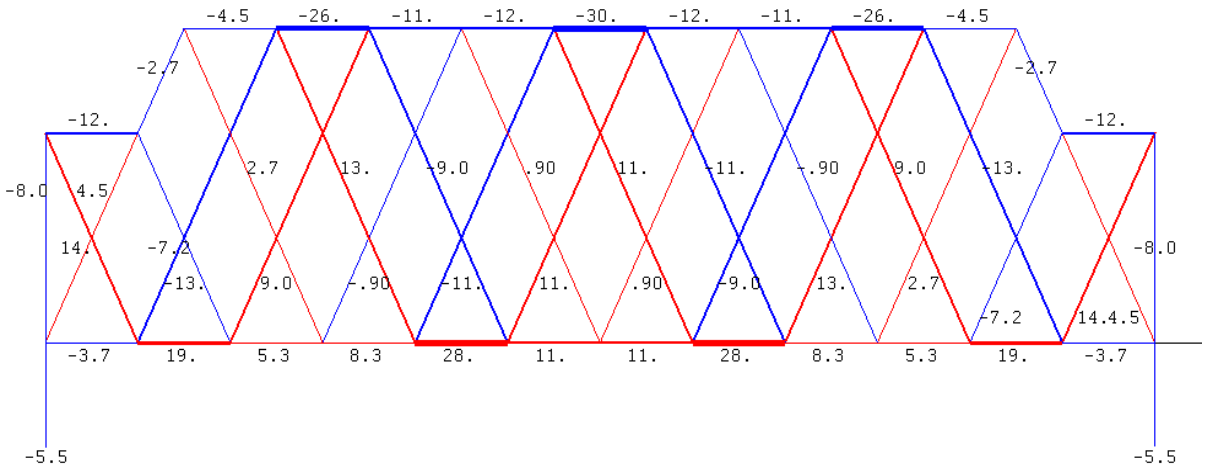

Fig. 5. Distribution of the efforts in case of an even load, $n = 5$, $k = 4$

The alternation of the magnitude of the force in the belts is typical for multi-lattice trusses under a uniform load. It should also be noted that the forces in some girder rods are comparable with the forces in the belts.

The distribution of forces in the truss rods under a concentrated force in the middle of the lower belt is shown in Fig. 6. Most of the rods in the girder are not loaded, the rods of the upper one are compressed, the efforts on the belt monotonously increase (along the module) from the edge to the middle. The lower belt is stretched.

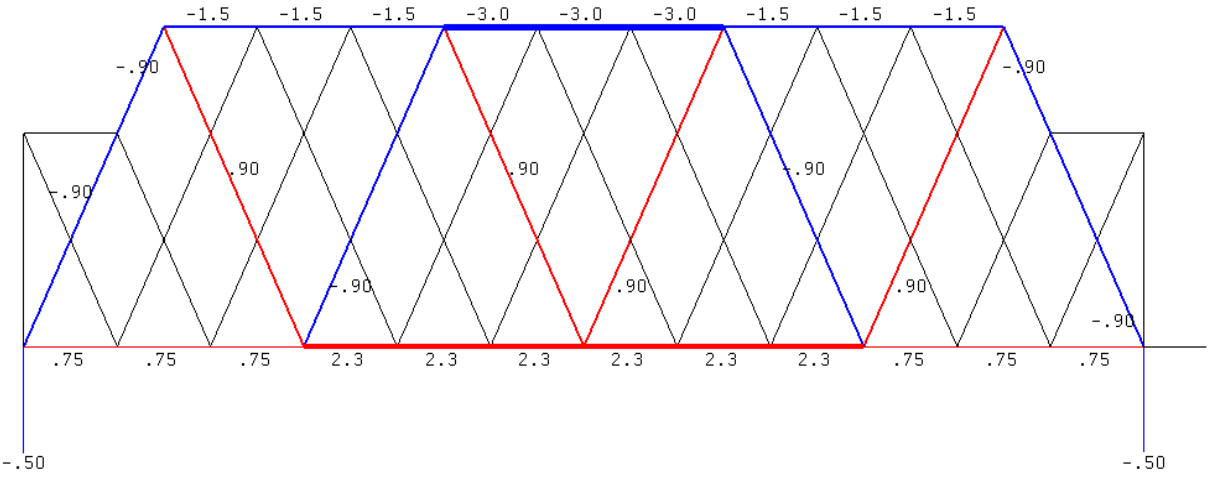

Fig. 6. Distribution of the efforts in case of an even load, $n = 5$, $k = 4$

In order to derive the formula for the dependence of the deflection on the number of panels, it is not necessary to write the forces in all the rods. The most interesting for the analysis of the stress state are the compressed rods in the middle of the upper and tensioned in the lower belt.

Let us write the expressions for the effort S_O (see Fig. 1) obtained based on the results of a static analysis of trusses with a successively increasing number of panels:

$$S_O = -4Pa/h, -7Pa/h, -15Pa/h, -20Pa/h, -32Pa/h, \ldots .$$

The recursive equation for this sequence obtained by the rgf_findrecur operator is as follows:

$$S_{O,k} = S_{O,k-1} + 2S_{O,k-2} - 2S_{O,k-3} - S_{O,k-4} + S_{O,k-5}.$$

The solution of the equation is as follows:

$$S_O = -Pa(6k^2 - 2((-1)^k - 9)k - 3(-1)^k + 3)/(8h).$$

Similarly, the dependence of the force in the middle rod of the lower belt on the number of panels is obtained:

$$S_U = Pa(6k^2 - 2((-1)^k + 3)k + (-1)^k - 5)/(8h).$$

3. Deflection and displacement of the support. In order to calculate the truss deflection (vertical displacement of node *C*) under a load, the Maxwell-Mohr formula is used:

$$\Delta = \sum_{j=1}^{n_s} N_j \bar{N}_j l_j / (EF), \quad (2)$$

where N_j are the efforts in the rod of the *j* girder caused by an applied load; $\bar{N}_j$ is the effort caused by a single force directed towards the original deflection and applied to the point *C*; l_j is the length of the rod; *EF* is the rigidity of the rods. Let us look at a load evenly distributed along the nodes of the lower belt.

Calculating the girders at k = 1, 2, 3,… sequentially, we get the solutions not depending on the number of panels:

$$= P\frac{C_{1,k}a^3 + C_{2,k}c^3 + C_{3,k}h^3}{EFh^2}. \quad (3)$$

The coefficients $C_{1,k}$, $C_{2,k}$, $C_{3,k}$ form sequences whose common members can be obtained in the *Maple* system. The coefficients $C_{1,k}$ at a^3 form the following numerical sequence whose first ten elements are: 18, 27/2, 411/2, 234, 984, 2403/2, 6267/2, 3816, 7794, 18675/2. In total, there must be 18 elements, i.e., it is necessary to sequentially calculate the deflection of 18 trusses in order to identify a common member of the sequence. This is found out as the *Maple* rgf_findrecur operator is being used to obtain a recursive equation that these numbers satisfy. A smaller sequence length does not yield an equation. The equation is of the ninth order:

$$C_{1,k} = C_{1,k-1} + 4C_{1,k-2} - 4C_{1,k-3} - 6C_{1,k-4} + 6C_{1,k-5} + 4C_{1,k-6} - 4C_{1,k-7} - C_{1,k-8} + C_{1,k-9}. \quad (4)$$

The solution of the equation is yielded by the rsolve operator:

$$C_{1,k} = (30k^4 + 20(1-(-1)^k)k^3 + 6(21-23(-1)^k)k^2 + 112(1-(-1)^k)k - 9(-1)^k + 9)/32.$$

The other coefficients are obtained in the identical way:

$$\begin{aligned} C_{2,k} &= (6(13-12(-1)^k)k^2 + (58-74(-1)^k)k - 3(-1)^k + 3)/16, \\ C_{3,k} &= k(1-(-1)^k). \end{aligned} \tag{5}$$

The solution (3) with the coefficients (4), (5) is the desired dependence in the case of a distributed load along the lower belt. While loading the middle of the span with a concentrated force (Fig. 1), the solution will have the form (3) with the coefficients:

$$\begin{aligned} C_{1,k} &= (4k^3 + 2(1-(-1)^k)k^2 + 2(12-11(-1)^k)k - 17(-1)^k + 17)/8, \\ C_{2,k} &= (6(5-4(-1)^k)k - 17(-1)^k + 17)/8, \\ C_{3,k} &= 1-(-1)^k. \end{aligned}$$

The order of recurrent equations for identifying the coefficients in this case is less, the coefficients themselves have a simpler form.

In order to identify the formula for the dependence of the displacement of the support on the number of panels, the above procedure must be repeated for the case of a unit force applied horizontally to the left support. Calculation according to the formula (1) for trusses with an increasing number of panels yields the following shift values:

$$\begin{aligned} &\delta_1 = 4Pa^2/(hEF),\ \delta_2 = 9Pa^2/(hEF),\ \delta_3 = 57Pa^2/(hEF), \\ &\delta_4 = 90Pa^2/(hEF),\ \delta_5 = 230Pa^2/(hEF),\ \ldots . \end{aligned}$$

Generalizing these solutions, we obtain the following dependence of the deflection on the number of panels:

$$\delta = a^2kP(6k^2 + 3(1-(-1)^k)k - 5(-1)^k - 1)/(4hEF).$$

It should be remembered that the actual number of panels n is expressed through the number k according to the formula (1).

4. Solution analysis. Let us look at the solution of the problem of truss deflection under the action of a uniform load along the lower belt (3) with the coefficients (4), (5). If we fix the value of the total load not depending on the number of panels, $P_0 = P(2n+1)$.

Let us design graphs depending on the number of dimensionless deflection panels

$$\Delta' = \Delta EF/(P_0L),$$

where the length of the span is denoted $L = 2(n+1)a = 100\,\text{m}$ (Fig. 7). The resulting dependence has a spasmodic character characteristic of lattice trusses. As the number of panels in-

creases, the magnitude of the jumps drops. The relative deflection of trusses having the same span but differing by only one panel can vary by several times. At $k = 3$, $h = 6$ m we have $\Delta' = 11.7$ and at $k = 4$ the deflection is three times smaller: $\Delta' = 3.5$. Using the formula (1) $n = 4$ and $n = 5$ correspond to the numbers $k = 3$ and $k = 4$ sec. This feature of the truss design makes one focus on the choice of the number of panels.

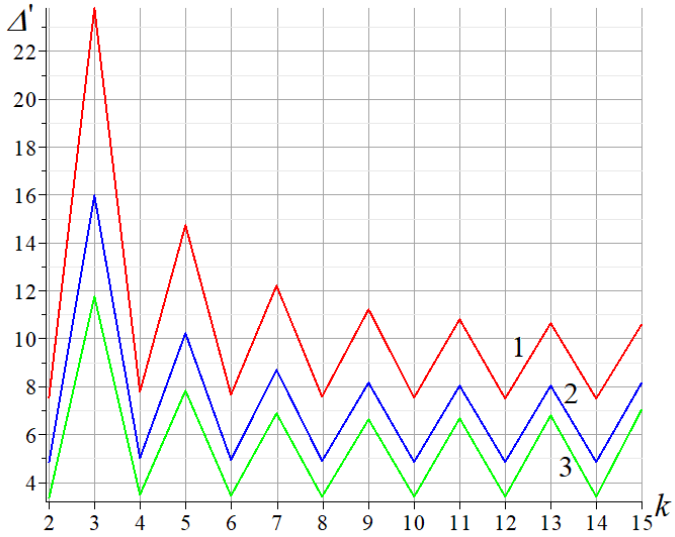

Fig. 7. Dependence of a relative deflection on the number of panels: 1— $h = 4$ m; 2— $h = 5$ m; 3 — $h = 6$ m

A dependence of a relative deflection of the support on the number of panels looks in another way (Fig. 8).

$$\delta' = \delta EF / (P_0 L).$$

The fluctuations in the shift value decrease with as the number of panels rises, and the curves go to the horizontal asymptote whose value can be identified using *Maple* tools:

$$\lim_{k \to \infty} \delta' = L / (18h).$$

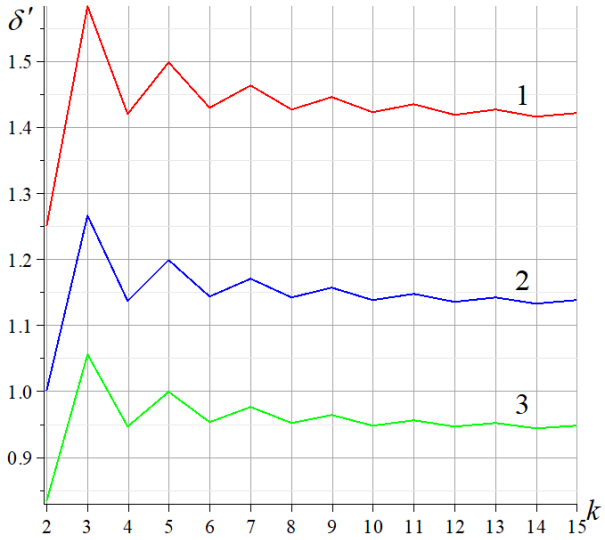

Fig. 8. Addiction displacement of the movable support on the number of panels: 1 — $h = 4$ m; 2 — $h = 5$ m; 3 — $h = 6$ m

Conclusions. The most distinct feature of the above scheme is its geometric variability manifesting itself when the number of panels is a multiple of three.

There were some mathematical difficulties in identifying he picture of possible speeds of units of a variable structure (in fact, an instantly variable mechanism). There is no relevant algorithm for addressing such a problem in kinematics. Thus a simple algorithm based on the calculation of a two-link was set forth. The only drawback of this algorithm is associated with the arbitrariness of the choice of the starting point for the subsequent chain of speed calculation graphs. A point was empirically identified in the middle of the lower belt whose speed was set to zero.

For a number of panels that are not a multiple of three, simple formulas for calculating the deflection and displacement of a support are obtained by induction, which are valid for any number of panels including those trusses where due to the huge size of the matrices of resolving equations numerical methods yield either a failure or a large error. It is thus convenient to use such analytical solutions both for the initial calculation of the designed structure and for evaluating numerical solutions.

References

1. Buka-Vaivade K., Kirsanov M. N., Serdyuk D. O. Calculation of deformations of a cantileverframe planar truss model with an arbitrary number of panels [Calculation of deformations of a cantileverframe planar truss model with an arbitrary number of panels]. *Vestnik MGSU*, 2020, vol. 15, iss. 4, pp. 510—517. DOI: 10.22227/1997-0935.2020.4.510-517.

2. Galishnikova V. V., Ignat'ev V. A. *Regulyarnye sterzhnevye sistemy. Teoriya i metody rascheta* [Regular rod systems. Theory and methods of calculation]. Volgograd, VolgGASU Publ., 2006. 551 p.

3. Domanov E. V. Analiticheskaya zavisimost' progiba prostranstvennoi konsoli treugol'nogo pro-filya ot chisla panelei [Analytical dependence of the deflection of the spatial console of a triangular profile on the number of panels]. *Nauchnyi al'manakh*, 2016, no. 6—2 (19), pp. 214—217.

4. Ignat'ev V. A. *Raschet regulyarnykh sterzhnevykh sistem* [Calculation of regular rod systems]. Saratov, Saratovskoe vysshee voenno-khimicheskoe voennoe uchilishche Publ., 1973. 433 p.

5. Rybakov L. S. Lineinaya teoriya ploskogo prizmaticheskogo karkasa [Linear theory of a flat prismatic frame]. *Izvestiya RAN. Ser. Mekhanika tverdogo tela*, 2001, no. 4, pp. 106—118.

6. Rybakov L. S. Lineinaya teoriya ploskoi ortogonal'noi reshetki [Linear theory of a flat orthogonal lattice]. *Izvestiya RAN. Ser. Mekhanika tverdogo tela*, 1999, no. 4, pp. 174—189.

7. Tin'kov D. V. Sravnitel'nyi analiz analiticheskikh reshenii zadachi o progibe fermennykh konstruktsii [Comparative analysis of analytical solutions to the problem of deflection of truss structures]. *Inzhenerno-stroitel'nyi zhurnal*, 2015, no. 5, pp. 66—73.

8. Tin'kov D. V. Formuly dlya rascheta progiba vsparushennoi balochnoi raskosnoi fermy s proiz-vol'nym chislom panelei [Formulas for calculating the deflection of a collapsed beam diagonal truss with a free number of panels]. *Stroitel'naya mekhanika i konstruktsii*, 2016, vol. 2, no. 13 (13), pp. 10—14.

9. Ferrari R., Cocchetti G., Rizzi E. Effective iterative algorithm for the Limit Analysis of truss-frame structures by a kinematic approach. *Computers & Structures*, 2018, vol. 197, pp. 28—41. DOI: https://doi.org/10.1016/j.compstruc.2017.11.018.

10. Greene R. L. *Classical mechanics with Maple*. New York, Springer Science & Business Media, 2000. 173 p.

11. Guest S. D., Hutchinson J. W. On the determinacy of repetitive structures. *Journal of the Mechanics and Physics of Solids*, 2003, no. 51, pp. 383—391.
12. Hutchinson R. G., Fleck N. A. Microarchitectured cellular solids — the hunt for statically determinate periodic trusses. *ZAMM Z. Angew. Math. Mech*., 2005, no. 85, pp. 607—617. DOI: https://doi.org/10.1002/zamm.200410208.
13. Hutchinson R. G., Fleck N. A. The structural performance of the periodic truss. *J. Mech. Phys. Solids*., 2006, no. 54, pp. 756—782.
14. Ilyushin A. S. The formula for calculating the deflection of a compound externally statically indeterminate frame. *Stroitel'naya mekhanika i konstruktsii*, 2019, vol. 3, no. 22, pp. 29—38.
15. Kirsanov M., Serdjuks D., Buka-Vaivade K. Analytical Dependence of Deflection of the Lattice Truss on the Number of Panels. *Lecture Notes in Civil Engineering*, 2020, vol. 70, pp. 25—35.
16. Kirsanov M. N. Mathematical model of a four-segment statically determinate spatial truss. *Structural Mechanics and Structures*, 2020, vol. 3, no. 26, pp. 7—12.
17. Kirsanov M. N. The deflection of spatial coatings with periodic structure. *Magazine of Civil Engineering*, 2017, no. 8 (76), pp. 58—66.
18. Rakhmatulina A. R., Smirnova A. A. The dependence of the deflection of the arched truss loaded on the upper belt, on the number of panels. *Science Almanac*, 2017, no. 2—3, pp. 268—271.
19. Tinkov D. V., Safonov A. A. Design optimization of truss bridge structures of composite materials. *Journal of Machinery Manufacture and Reliability*, 2017, vol. 46, no. 1, pp. 46—52.
20. Vorobev O. Bilateral analytical estimation of first frequency of a plane truss. *Construction of Unique Buildings and Structures*, 2020, vol. 92, Article № 9204. DOI: 10.18720/CUBS.92.4.