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Analytical Expression of the Dependence of the Multi-lattice Truss Deflection on the Number of Panels

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Abstract:

The object of research is a flat statically determinate trapezoidal truss with a rectilinear lower chord and four supports, one of which is a pinned, and three are roller. The purpose of this work is to analyze the dependence of the deflection of the truss and the shift of the movable support on the size, load, and number of panels. The load concentrated in the middle of the span, the load uniformly distributed over the nodes of the upper or lower belt are considered. **Method.** The initial forces in the elements are determined in analytical form by method of joints in the Maple computer mathematics system. The dependence of the truss performance characteristics on the number of panels is derived by induction based on analytical calculations of the sequence of trusses with different numbers of panels. External static uncertainty is revealed by adding five reactions of supports to the number of unknown components of the equilibrium system of the structure. The deflection of the truss and the displacement of the support are based on the Maxwell-Mohr formula. **Results**. By solving a number of problems for trusses with a different number of panels, it is found that for trusses whose number of panels is a multiple of three, the determinant of the system of equilibrium equations of nodes turns to zero, which corresponds to the instantaneous kinematic variability of the truss. The corresponding scheme of possible node speeds was found. For kinematically unchangeable trusses, formulas for deflection depending on the number of panels are obtained. The coefficients in the formula are polynomial type. The solution graphs show an abrupt increase in deflection as the number of panels increases.

1 Introduction

The practice of real calculations of building structures is based, as a rule, on numerical modeling of the object and the use of standard computer programs, most of which use the finite element method [1-7]. With the advent of computer mathematics systems, analytical methods began to develop [8-13]. Separately, we can distinguish analytical methods that lead to relatively compact resolving formulas that include not only the size, load value, and material properties of elements, but also quantitative characteristics, such as the number of periodicity elements in a regular structure. In a truss, this characteristic is the number of panels or the number of elements (rods). The handbook [14] contains analytical solutions for the deflection problem of more than 70 planar statically definable trusses with various lattices, chord outlines, and supports, including externally statically indeterminate trusses. Solutions are obtained for three types of loads (concentrated force in the middle of the span, uniformly distributed load along the lower or upper chord). The Maple system of computer mathematics and the method of induction served as a tool for obtaining solutions in a closed form. Despite the large number of problems already solved [15-19], the search for regular statically definable trusses for which the induction method can be applied and an analytical solution obtained continues. Hutchinson R. G., Fleck N. A., and Zok F. W., Latture R. M., Begley M. R. were among the first to begin such searches, not only for flat but also for spatial schemes [20–22]. In this paper, we propose a new scheme for a multi-lattice statically definable truss and derive formulas for its deflection under various loads.

2 Methods

The scheme of the truss. **Determination of the forces in the bars**. The proposed scheme cannot be fully attributed to regular constructions. The frequency of panels is broken by beveled sections at its ends. Taking the struts and a rod of length a in the lower chord as one panel, we assume that the truss consists of 2*n* panels (Fig. 1-3).

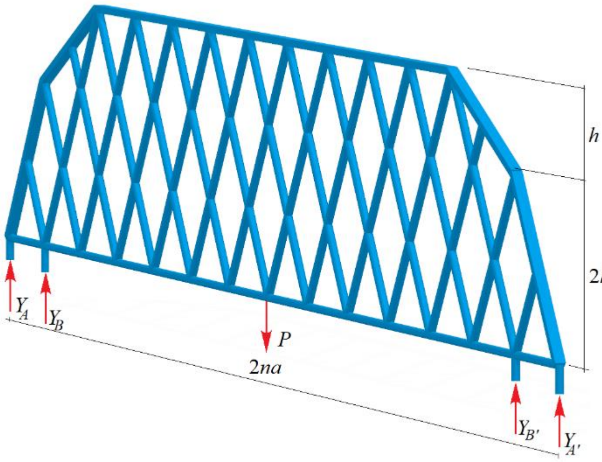

Fig. 1. Truss. Concentrated load in the middle of the span, n=7

Four external supports with five unknown reactions turn the truss into an externally statically indeterminate structure. The reactions of the supports cannot be found separately from the forces in the rods.

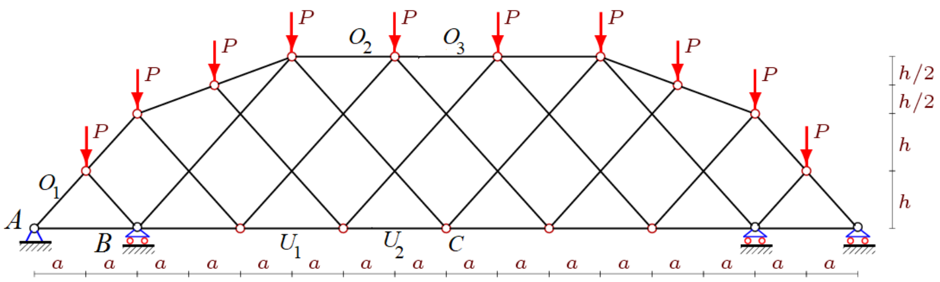

Fig. 2. Truss. The load on the upper chord, *n*=4

In addition, such a scheme does not allow the use of the section method to calculate the initial forces in almost all the rods (except for a few along the edges of the structure). However, if we compile a general system of equilibrium equations for the nodes of the truss, including five external reactions among the unknowns, the problem of determining the reactions of supports and initial forces can be solved, since the number of rods in the truss and the number of reactions total $m=8n+6$ two times the number of nodes. To compile the matrix of the system of equilibrium equations, we use a program written in Maple. The rods and nodes of the truss are numbered (Fig. 4), the coordinates of the nodes and the lattice structure are entered into the program [7-9].

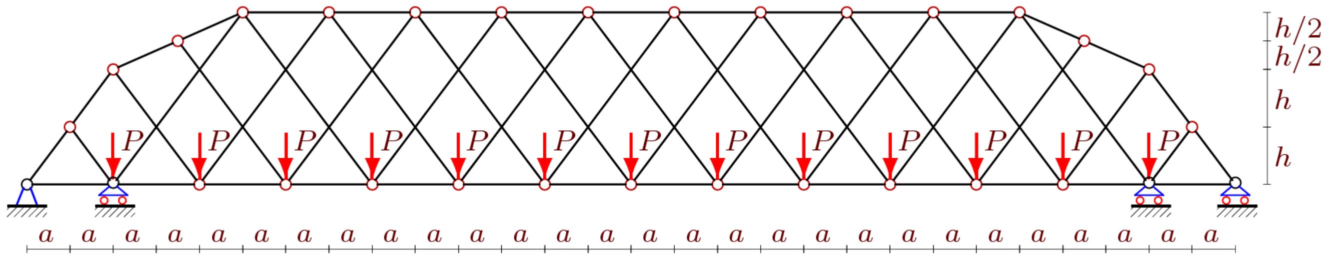

Fig. 3. Truss. The load on the bottom chord, *n* = 7

It is convenient to compile the system of equations in a program written in Maple and tested in various problems with trusses [14-18].

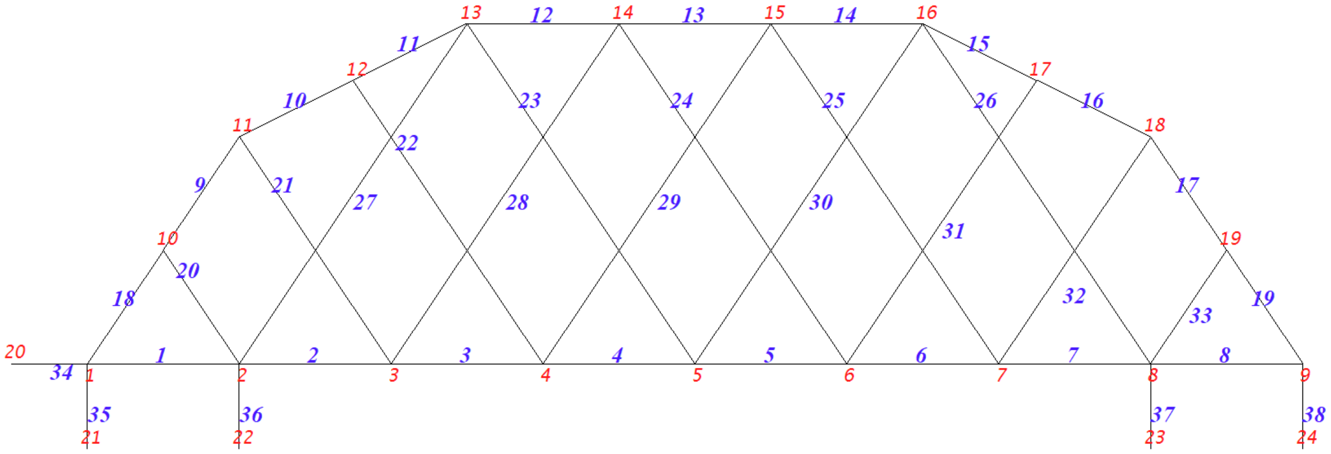

Fig. 4. The numbers of nodes and rods of the truss, $n = 4$.

The coordinates of the nodes of the lower zone, for example, have the form $x_i = 2a(i-1), y_i = 0,\ i = 1,...,2n.$ The lattice structure is also defined as a graph in discrete mathematics by the numbers of nodes (vertices of the graph) at the ends of the rod. Special lists of bars (graph edges) are introduced. Chord rods have the following lists:

$$N_i = [i, i+1],\ i = 1,...,2n,$$

$$N_{i+2n} = [i+2n+1, i+2n+2],\ i = 1,...,2n+1.$$

The matrix $\mathbf{G}$ of equations of equilibrium $\mathbf{GS} = \mathbf{B}$, where $\mathbf{S}$ is the vector of unknowns of length *m*, $\mathbf{B}$ is the vector of loads, consists of direction cosines of forces. The solution of the system $\mathbf{S} = \mathbf{G}^{-1}\mathbf{B}$ gives the forces in the rods in a symbolic form. The inverse matrix $\mathbf{G}^{-1}$ is found by the methods of the Maple system.

The distribution of forces in the rods under the action of the load along the upper chord can also be obtained using the graphic capabilities of Maple (Fig. 5).

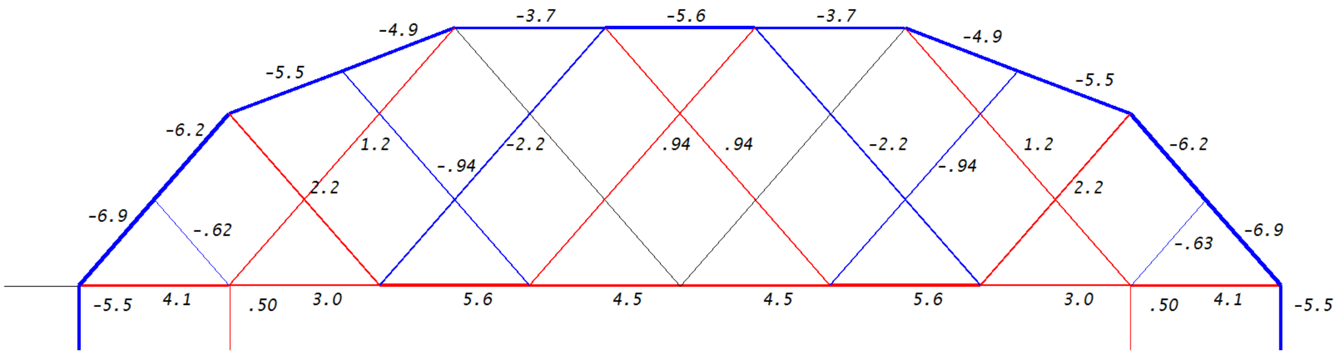

Fig. 5. The distribution of forces. Top chord load, $n = 4,\ a = 3$ m, $h = 4$ m.

Compressed rods are highlighted in blue, rods subjected to the tension in red. The thickness of the segments is proportional to the moduli of forces referred to the magnitude of the load *P*. Unloaded rods are indicated by the thinnest black lines. The most compressed are the rods of the upper chord at the ends of the truss, and the most extended rods are not in the middle of the span, but near the middle rods. Such a nonmonotonic change in the forces in the chords is characteristic of multilattice trusses [22]. A similar distribution of initial forces when loading the lower chord (Fig. 3) is shown in Fig. 6.

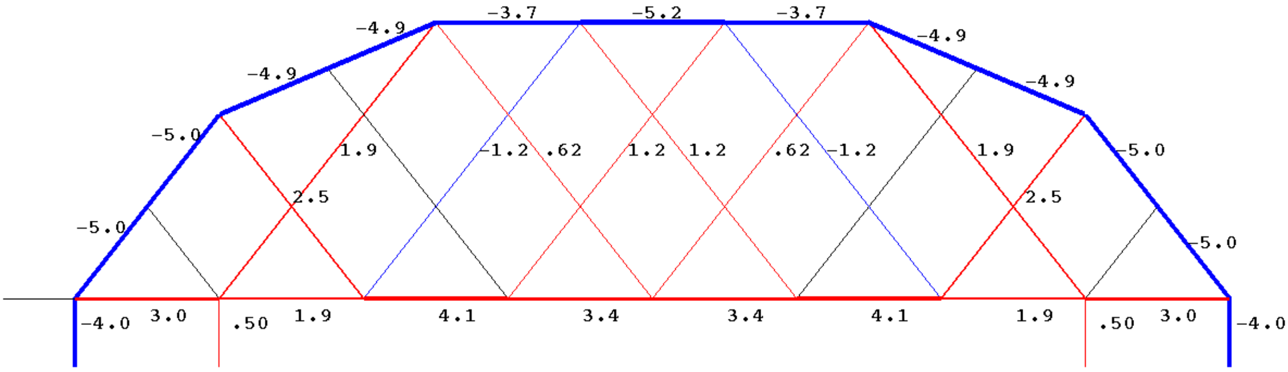

Fig. 6. The distribution of forces. Lower chord load, $n = 4,\ a = 3$ m, $h = 4$ m.

In this case, the most compressed rod was in the upper zone in the middle of the span. It is also interesting to note that the extreme support rods are compressed, and the adjacent ones are extended. These bonds do not support the structure from below but keep it from shifting up. The force diagram clearly shows that the core rods are mainly involved in the work of the structure. Most of the lattice rods are subjected to the tension, four rods at *n* = 4 are not loaded, two rods are compressed with little force. All this makes it possible to use small cross-section rods for the lattice.

The case of kinematic degeneracy. Despite the comparative simplicity of the scheme, the truss has one unexpected dangerous property. When the number of panels is a multiple of three, the design turns into an instantly variable mechanism. This was discovered in the process of analytical calculation of efforts with a different number of panels. For *n* = 3*k*, *k* = 1,2,3... the determinant of the matrix vanishes. Note that in numerical calculations this is not immediately noticeable, the calculation error hides this effect, which makes it especially dangerous for engineers who rely entirely on the infallibility of specialized numerical calculation programs. The distribution pattern of possible node velocities (Fig. 7) confirms this case. When *n* = 3, three nodes are displaced, others remain motionless. There is a connection between the speeds of the nodes. The instantaneous center of velocity of the rod 4-10 is at point *K*. Hence, we have $4u/d = v/a$ where $d = \sqrt{9a^2 + h^2}$.

One of the signs of the possibility of kinematic degeneration of the circuit is the presence in the truss of a chain of rods with «zero» rods at the ends and «special» intermediate nodes. Here, “zero” rods are rods in which the force is zero according to the zero rod lemma. The lemma states that *if three rods are attached to an unloaded node, two of which are on the same straight line, then the force in the third is zero*. For *n* = 3, these are rods 10–4 and 4–13, which are part of the chain 10–4–13 (Fig. 7). By «special» nodes, we mean nodes for which an equilibrium equation can be written that contains only the forces in the rods of the chain. In fact, this is a node connected to four rods, two of which are the rods of the chain, and the other two are not included in it. The proof of the health of the observed symptom follows from the equilibrium equations of the nodes 10, 4, and 13. If, for example, the node 10 is not loaded, then the rod 10-4 has a force equal to zero. Then, from the equilibrium equation of the “special” unit 4 in the projection onto the vertical, the force in the rod 4-13 should also be zero. On the other hand, if an external force acts on the node 13, then the rod 4-13 must have a force depending on the load. This creates a static contradiction, which manifests itself in the fact that the determinant of the system of equations of equilibrium of nodes vanishes. In the truss under consideration, the number of rods of which is not a multiple of three (Fig. 4), a similar chain 12-4-15-7-18 has only a “zero” rod 12-4 at one end, the condition of the lemma is not performed at the other end (node 18). Nodes 4, 15, and 7 are “special”.

Thus, this feature allows you to identify a hidden design defect, without resorting to calculation.

In [23,24], multi-lattice trusses are described in which kinematic degeneracy is also manifested when the number of panels is a multiple of three. The case of kinematic degeneration of a truss for a number of panels was noted in [25]. More than 20 schemes of virtual velocities for planar statically determinate trusses of various designs that allow kinematic variability for a certain number of panels are given in the handbook [14].

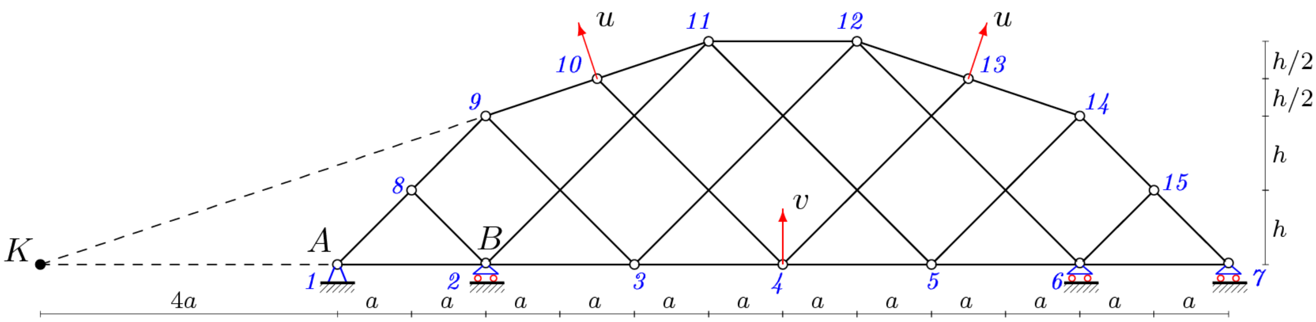

Fig. 7. Scheme of possible velocity of nodes, *n* = 3

Excluding variable trusses from consideration, i.e., numbers that are multiples of three, we will carry out all further calculations for $n=(6k-(-1)^k-3)/4,\ k=3,4,5,...$.

Formula for deflection. Assuming that the stiffness of the lattice rods and the chords are different, to calculate the deflection, we use the Maxwell-Mohr's formula in the form:

$$\Delta=\sum_{j=1}^{4n+3}\frac{S_j s_j l_j}{EF}+\gamma\sum_{j=4n+4}^{m-5}\frac{S_j s_j l_j}{EF}, \qquad (1)$$

where EF is the stiffness of the rods of the chords, EF/γ is the stiffness of the rods of the lattice, S_j is the force in the rod *j* of the truss from the action of the applied load, s_j is the force from the unit force applied to the node *C* in the middle of the lower chord, l_j is the length of the rod *j*. Five support rods, adopted rigid, are not included in the amount. Calculations show that regardless of the number *k*, the shape of the solution for deflection does not change:

$$\Delta_k=\frac{P(C_1a^3+(C_2+\gamma C_4)c^3+C_3d^3)}{EFh^2}, \qquad (2)$$

where $c=\sqrt{a^2+h^2}$. The coefficients $C_i,\ i=1,..,4$ depend only on *k*. To determine these dependencies, we use the induction method. To determine the total member of the sequence of coefficients C_1, it was necessary to calculate 18 trusses. The recurrence equation, which is satisfied by the common term in the sequence 0, 11, 70, 269, 768, 1667, 3366, 5813, 9856, 15035, 22950, 32381, 46080, 61619, 83398, 107237, 139776, 174443, is given by the operator rgf_findrecur of the Maple system. The ninth order equation has the form:

$$C_{1,k}=C_{1,k-1}+4C_{1,k-2}-4C_{1,k-3}-6C_{1,k-4}+6C_{1,k-5}+4C_{1,k-6}-4C_{1,k-7}-C_{1,k-8}+C_{1,k-9}.$$

The solution of this equation is given by the resolve operator:

$$C_1=(30k^4-4((-1)^k+15)k^3+2(7(-1)^k+17)k^2-4(5(-1)^k+3)k+39+41(-1)^k)/16. \qquad (3)$$

Other coefficients are also obtained in the case of a load in the upper zone:

$$C_2=((-1)^k+1)(4k-1)/2,$$
$$C_3=((-1)^k+1)(4k-5)/16, \qquad (4)$$
$$C_4=(6k^2+4((-1)^k-8)k+25-5(-1)^k)/8.$$

For the load in the lower zone (Fig. 3), the form of solution (2) does not change, and the coefficients have the form:

$$C_1=(30k^4-4((-1)^k+15)k^3+2(11(-1)^k-1)k^2+32(1-2(-1)^k)k+9+55(-1)^k)/16,$$
$$C_2=2(1+(-1)^k)(k-1),$$
$$C_3=(1+(-1)^k)(k-1)/4, \qquad (5)$$
$$C_4=(6k^2+4((-1)^k-8)k+11(-1)^k+41)/8.$$

The corresponding coefficients of formula (2) in the case of the action of a concentrated force in the node *C* (Fig. 1) have the form:

$$C_1=(4k^3+6((-1)^k-1)k^2+2(4-3(-1)^k)k-15-9(-1)^k)/4,$$
$$C_2=2+2(-1)^k,\ \ C_3=(1+(-1)^k)/4,\ C_4=(3k-2+(-1)^k)/2. \tag{6}$$

Offset of the support. The calculation of the rigidity of the trusses, as a rule, comes down to the calculation of the deflection of the structure under the action of various loads. However, the horizontal displacement of the movable support is also a rather important operational characteristic of the structure. In this truss, there are three movable supports. The most interesting is the displacement of the extreme support of the symmetric support *A*. To calculate the horizontal displacement in formula (1), the forces must be calculated from the action of a single dimensionless horizontal force applied to the movable support. As a result of the generalization of 16 solutions for trusses with a successively increasing number of panels, starting from k = 3, for the load along the upper chord at $\gamma=1$, we obtain:

$$\delta=Pa^2(6k^3-(9+7(-1)^k)k^2+(25+9(-1)^k)k-14-6(-1)^k)/(2hEF).$$

Similar solutions for the load on the nodes of the lower chord the formula has the form:

$$\delta=Pa^2(6k^3-(9+7(-1)^k)k^2+(7+11(-1)^k)k-4(-1)^k-4)/(2h).$$

The horizontal displacement of the movable support from the action of the concentrated load P on the hinge C depending on the number of panels has the form:

$$\delta=3Pa^2(2k^2+2((-1)^k-1)k+1-(-1)^k)/(4hEF).$$

Note that, as in the case of deflection, the order of the polynomial in k for a concentrated load is one less than for a distributed load.

Initial forces in the rods and the reaction of the supports. Simultaneously with the output of formulas for the deflection dependence on the number of panels, you can get formulas for the initial forces in some of the most dangerous in relation to the loss of strength or stability of the rods and the reaction of the supports depending on the number of panels. For the upper chord load we have the following formulas:

$$O_1=-Pc((-1)^k-1+6k-2(-1)^k k)/(4h),$$
$$O_2=-Pa(6k^2+(-1)^k-2(-1)^k k-6k-1)/(8h),$$
$$O_3=-Pa(6k^2-(10(-1)^k+6)k+3+9(-1)^k)/(8h),$$
$$U_1=Pa(6k^2-(10(-1)^k+6)k+3+9(-1)^k)/(8h),$$
$$U_2=Pa(6k^2-(2(-1)^k+6)k+(-1)^k+7)/(8h),$$
$$Y_A=P(2(3-(-1)^k)k+(-1)^k-1)/4,$$
$$Y_B=P(1-(-1)^k(1-k))/2.$$

Obviously, critical in terms of stability loss in this design will be the rods O_1, O_2 in the middle of the upper chord, in which the dependence of initial forces on the number of panels is quadratic. The values of forces in them can alternate due to terms with a variable sign $(-1)^k$. The same applies to the rods subjected to the tension of the lower chord. It is interesting to note that the reaction of support B can change the sign depending on the parity of the number of panels. Thus, in some cases, this support really supports the structure from below, in others, on the contrary, it keeps it from breaking off. It is assumed that the support is two-sided, for example, a hinge rod. Support *A*, whose reaction is always positive, can be a roller (one-way connection).

3 Results and Discussion

Solution (2) with coefficients (3-6) for various types of loads allows not only to obtain the deflection value for any number of panels without losing the accuracy of the result, but also to analyze the nature of the change in stiffness when changing the design parameters. This can be most clearly seen in the graphs constructed with some specific data. We introduce a dimensionless deflection $\Delta'=\Delta EF/(P_sL)$, where $P_s=P(2n+2)$ — the total load in the case of loading the upper chord, $a=L/(2n)$ — the length of the panel at a fixed span. The dependence of the deflection on the number of panels turns out to be significantly non-monotonic (Fig. 8). The jumps at the beginning of the chart are especially large. As k increases, the dependence becomes smoother. In comparison with similar curves constructed also from analytical solutions for arch and beam trusses [23-25], there is no noticeable increase in deflection here. Within reasonable limits of changing the number of panels, the relative deflection is almost unchanged. As before, a constant span independent of the number of panels and a constant total load is accepted. The amplitude of the deflection change for close values of *k* varies up to 100% at low altitudes (*h*=2m). As the height of the truss increases, the deflection jumps are less. The curves are constructed for the case of rods of the same stiffness ($\gamma = 1$). The effect of this parameter on the result is insignificant. The main load is borne by the rods of the chords.

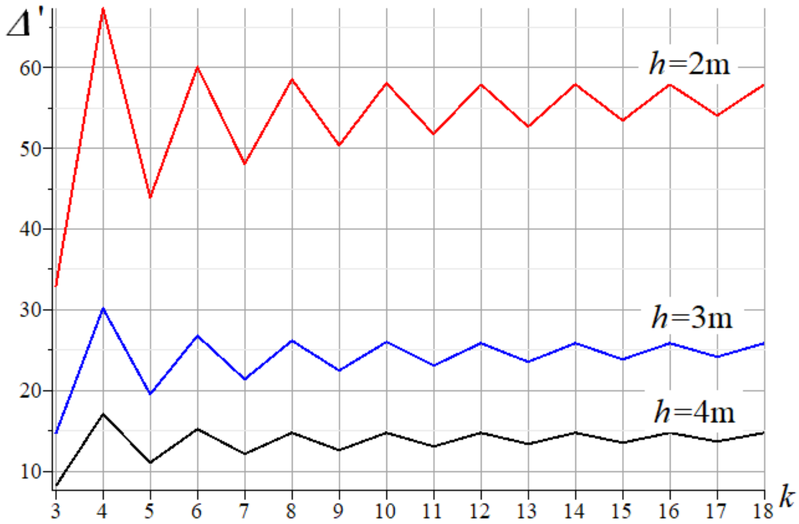

Fig. 8. The dependence of the deflection on the number of panels, at $\gamma = 1$, $L = 100$ m.

The dependence of the deflection on the height of the truss was predictable. As the height *h* increases, the deflection within certain limits of the change in *h* decreases monotonously (Fig. 9). However, the study of the asymptotics of this solution shows that the curves have linear asymptotics that depend on the number *k.* We have a limit

$$\lim_{h\to\infty} \Delta'/h = \alpha, \quad \alpha = \frac{28k - 12k^2 - 44(-1)^k k + 23(-1)^k - 37}{8L((-1)^k - 1 - 6k)}.$$

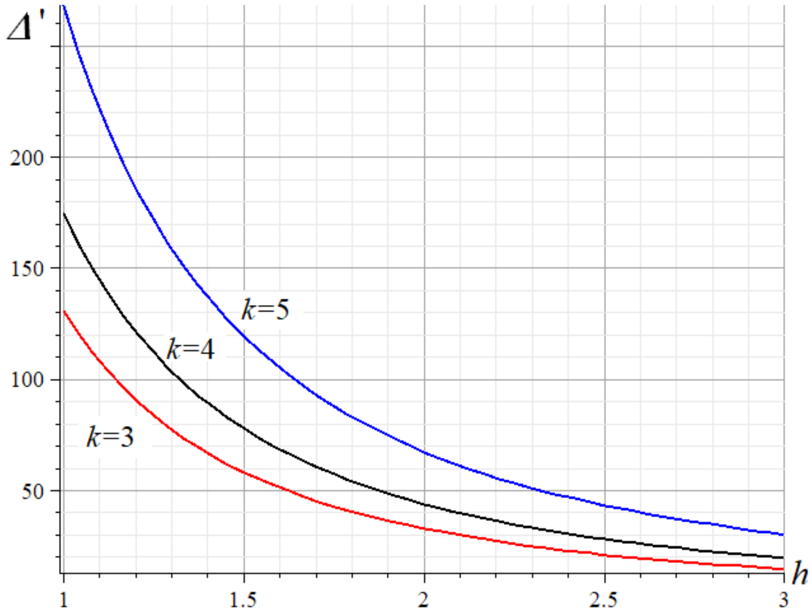

Fig. 9. Dependence of the deflection on the height *h* at $\gamma = 1$, $L = 100$ m.

However, only for *k*=3 the angle of inclination is negative $\alpha = -3/(10L)$. This means that, for *k* > 3, the curves have a minimum. In General, this minimum means that the problem can be optimized for rigidity by selecting the appropriate height value. Here, the numerical values of the extremum are such that the proportions of the optimal design are practically impossible to implement: the height of the optimal truss is greater than the span length.

4 Conclusions

1. The proposed truss scheme allows for an exact solution of the problem of the dependence of the deflection of the structure and the shift of the movable support on the number of panels for several types of loads. The final formula has a polynomial form depending on the number of panels.

2. The property of instantaneous changeability of the scheme for a certain number of panels is detected. A feature is proposed that reveals the instantaneous changeability of the scheme by the presence of a chain of rods of a certain type.

3. Diagrams of the distribution of forces on the truss rods, constructed using analytical formulas of the solution, showed the features of the stress state of the multi-lattice structure under various loads.

4. The asymptotic of deflection by the value of the height of the truss is found, indicating a theoretically possible, but not practically feasible, minimum deflection.

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