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Formulas for Fundamental Natural Frequency of Plane Periodic Truss

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Conflicts of interest
The author declares that there is no conflict of interest.

Abstract. This study considers a plane statically determinate truss with double lattice structure and without a lower chord. Well-known versions of this design are Fink and Bollman trusses. Two methods are used to derive the analytical relationship of the lower limit of the fundamental frequency with the number of panels in the periodic structure. It is assumed that mass of the truss is concentrated at its joints (nodes). The nodes vibrate vertically, and the number of degrees of freedom coincides with the number of nodes. The stiffness analysis of the truss is performed using the Maxwell — Mohr method. The forces in the elastic elements and the reactions of the roller and pin supports are calculated by the method of joints depending on the size of the truss and its order of periodicity. The system of linear equations is solved using the inverse matrix method. The Dunkerley method of partial frequencies is used to calculate the lower limit of the fundamental frequency. For a series of solutions obtained for trusses with different number of panels, the common term in the sequence of solution formulas is found by induction using Maple software. The solution coefficients have polynomial form in the number of panels of order not higher than the fifth. The solution is compared with an approximate version of the Dunkerley method, in which the sum of the terms corresponding to partial frequencies is calculated using the mean value theorem. The closeness of the frequency obtained by the two analytical methods to the numerical frequency spectrum solution is shown by particular examples. An approximate version of the Dunkerley method has a simpler form and an accuracy comparable to the original Dunkerley method.

Keywords: plane truss, Dunkerley method, fundamental frequency, analytical solution, natural vibration, periodic structure, induction method, Maxwell — Mohr method

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Формулы для основной частоты собственных колебаний плоской регулярной фермы

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Аннотация. Рассмотрена модель плоской статически определимой фермы решетчатого типа без нижнего пояса с двойной решеткой. Известные аналоги такой конструкции — ферма Финка и ферма Больмана. Двумя методами выводится аналитическая зависимость нижней границы основной собственной частоты регулярной конструкции от числа панелей. Предполагается, что его масса фермы сконцентрирована в ее узлах. Узлы совершают колебательные движения по вертикали, число степеней свободы совпадает с числом узлов. Расчет жесткости фермы производится с помощью интеграла Максвелла — Мора. Усилия в упругих стержнях и реакции подвижной и неподвижной опор вычисляются методом вырезания узлов в зависимости от размеров фермы и ее порядка регулярности. Система линейных уравнений решается с помощью метода обратной матрицы. Для расчета нижней границы основной частоты используется метод парциальных частот Донкерлея. Для серии решений, полученных для ферм с различным числом панелей, методом индукции в системе символьной математики Maple находится общий член последовательности расчетных формул. Коэффициенты формулы имеют вид полиномов по числу панелей порядка не выше пятого. Решение сравнивается с приближенным вариантом метода Донкерлея, в котором сумма слагаемых, соответствующих парциальных частотам, вычисляется по теореме о среднем. На конкретных примерах показана близость частоты, полученной двумя аналитическими методами, численному решению задачи о спектре частот. Приближенный вариант метода Донкерлея имеет более простую форму и точность, сопоставимую с исходным методом Донкерлея.

Ключевые слова: пространственная ферма, метод Донкерлея, основная частота, аналитическое решение, собственные колебания, регулярная конструкция, спектр, спектральная константа, метод индукции, формула Максвелла — Мора

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1. Introduction

The first (fundamental) natural frequency of vibration is one of the main dynamic characteristics of a structure. To calculate this frequency in practice, the finite element method [1–3], which is implemented in standard engineering software, is used most often. Thus, it is possible to calculate the spectrum of all natural frequencies of a structure. A wide range of trusses can be calculated with numerical methods: statically indeterminate structures, structures with various types of fastenings, material inhomogeneities, errors in manufacture and installation of their elements. Analytical solutions for the lower and upper bounds of the fundamental frequency exist for simple statically determined trusses. Such solutions are especially valuable for periodic structures. One of the

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methods for obtaining analytical solutions is the method of induction [4–6]. Calculations of deflections and displacements of roller supports of plane periodic trusses with an arbitrary number of panels were performed using Maple software in [7; 8]. Deflections of space trusses depending on the order of periodicity (number of panels) are calculated in [9; 10]. R.G. Hutchinson and N.A. Fleck were the first who raised the question of the existence of statically determinate truss structures [11; 12]. The same issues were explored by F.W. Zok, R.M. Latture, and M.R. Begley [13]. A. Kaveh [14; 15] solved the problems of optimization and classification of various plane and space periodic trusses. The Maple mathematical system was used in [16; 17] for calculating elements of building structures in analytical form. Handbooks [18; 19] contain formulas for calculating deflections of various statically determinate plane trusses with an arbitrary number of panels. The formula for the lower estimate of natural vibrations of a planar periodic trussed beam with a rectilinear upper chord was derived in [20]. The method of induction was applied in [21] to obtain the equation for relationship between the deflection of a plane truss and the order of periodicity. A version of the Dunkerley method for estimating the fundamental frequency of a plane truss as applied to periodic structures was proposed in [22]. Simplification of the solution formula for the fundamental frequency is achieved here by calculating the sum of partial frequencies using the mean value theorem.

In this paper, a design of a statically determinate lattice truss is proposed, and an analytical relationship of the fundamental frequency with the dimensions, weight, and number of panels of the truss is derived. The numerical solution and the solution according to the simplified Dunkerley method [22] are compared.

2. Structure

A planar statically determinate truss consists of $2n$ panels of length a and height $2h$ (Figure 1). Four diagonal elements connected to nodes 8 and 16 are $c=\sqrt{a^2+h^2}$ long, the other diagonal elements are $2c$ long. An important feature of the truss is the absence of the lower chord. The Bollman [23] and Fink [24] plane trusses have a similar structure without a lower chord.

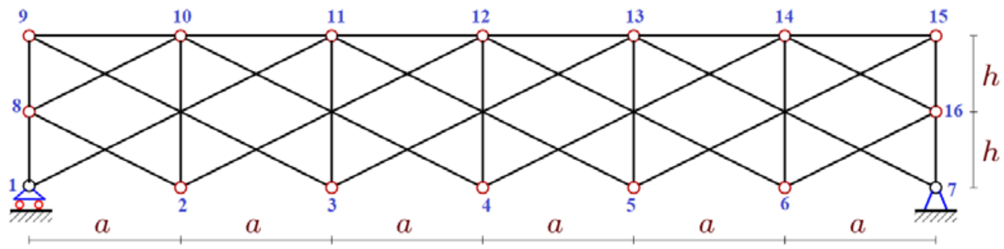

Figure 1. Truss structure, n = 3
Source: made by the author

The truss model consists of $\eta=9n+3$ elements, including three elements corresponding to the roller and pin supports. The number of nodes in which the entire mass of the truss is concentrated is $K = 4(n + 1)$. It is assumed that the masses vibrate along the vertical y-axis. The number of degrees of freedom of the truss, according to this assumption, is equal to K. The truss hinges are ideal and the material of the elements is elastic.

3. Method

3.1. Analysis of Forces. The analysis of forces in the truss elements is performed in symbolic form with Maple software [8]. The algorithm of this program is used in [3–7; 9–10; 20–24]. The hinge coordinates are entered in loops with a parametrically defined length. The coordinate origin is located at the left roller support:

$$x_i = x_{2n+2+i} = a(i-1),\ y_i = 0,\ y_{2n+2+i} = 2h,$$
$$x_{2n+2} = 0,\ y_{2n+2} = h,\ x_{4n+4} = 2na,\ y_{4n+4} = h,\ i = 1,..,2n+1.$$

The lattice structure is determined by the order in which the elements are connected to nodes (hinges). The order is given by lists containing the indices of the ends of the corresponding elements $\Phi_i, i = 1,\ldots,\eta$ Ascending diagonal elements, for example, are defined by the following lists: $\Phi_i=[i,i+2n+4],\quad i=1,..,\ 2n-1.$

The equilibrium equations for all nodes, including the support ones, are written in matrix form: $\mathbf{GS}=\mathbf{T}$, where $\mathbf{G}$ is the matrix of directional cosines of the internal forces in the elements, $\mathbf{S}$ is the vector of unknown forces and reactions of the supports, $\mathbf{T}$ is the load vector. The directional cosines of the forces are calculated from the coordinates of the nodes and values $\Phi_i, i=1,..,\eta$, which provide the order of connection of the elements to nodes. Horizontal and vertical nodal loads applied to node *i* are written, respectively, into odd T_{2i-1} and even T_{2i} coefficients of this vector. The matrix equation is solved in symbolic form in Maple using the inverse matrix method: $\mathbf{S}=\mathbf{G}^{-1}\mathbf{T}$.

3.2. Dunkerley Method for Estimating Fundamental Frequency. In cases where only the first (fundamental) frequency is required to analyze the structure, the Dunkerley lower estimate or the Rayleigh upper estimate can be used to calculate it. The latter estimate is more accurate, however, its analytical expression is very cumbersome.

The approximate lower estimate of the fundamental frequency according to Dunkerley is:

$$\omega_D^{-2}=\sum_{i=1}^{K}\omega_i^{-2}, \tag{1}$$

where ω_i are the partial frequencies calculated for each mass separately.

The equation of mass vibration at node *i*:

$$m\ddot{y}_i+d_i y_i=0,\ \ i=1,2,...,K. \tag{2}$$

Stiffness coefficient d_i is the reciprocal of the flexibility coefficient, which is determined using the Maxwell — Mohr method formula. The summation in this formula is performed over all truss elements, including three elements that substitute supports:

$$\delta_i=1/d_i=\sum_{j=1}^{\eta}\left(S_j^{(i)}\right)^2 l_j/(EF). \tag{3}$$

From (1) it follows that $\omega_i=\sqrt{d_i/m}$. From this, the formula for the Dunkerley frequency is obtained:

$$\omega_D^{-2}=m\sum_{i=1}^{K}\delta_i=m\Delta_n. \tag{4}$$

4. Results and Discussion

Variable δ_i included in the final formula for frequency (4) is in fact the vertical displacement of node *i* due to a unit vertical force applied to this node. Factor Δ_n is the sum of all such displacements for a truss of order *n*. To derive the relationship of Δ_n with the number of panels, the method of induction is used. The calculations give the following sequence:

$$\Delta_1=(c^3+30h^3)/(2h^2EF),$$
$$\Delta_2=(8a^3+9c^3+50h^3)/(2h^2EF),$$
$$\Delta_3=(448a^3+329c^3+894h^3)/(18h^2EF),$$
$$\Delta_4=(168a^3+105c^3+220h^3)/(2h^2EF),$$
$$\Delta_5=3(704a^3+407c^3+786h^3)/(10h^2EF),...$$

When finding a common term in this sequence, it was necessary to calculate ten trusses. Maple operators for a sequence of smaller lengths do not find a pattern. As a result, the following solution is obtained:

$$\Delta_n=(C_1a^3+C_2c^3+C_3h^3)/(h^2EF), \tag{5}$$

where the coefficients have the following form:

$$C_1 = 4(n^2-1)(4n^2-1)/45,$$
$$C_2 = (4n^2-1)(4n^2+11)/90, \tag{6}$$
$$C_3 = (16n^5-80n^3+480n^2+199n+60)/(45n).$$

As a result, the lower estimate of the fundamental frequency according to Dunkerley has the form:

$$\omega_D = h\sqrt{\frac{EF}{m(C_1a^3+C_2c^3+C_3h^3)}}\ . \tag{7}$$

The considered algorithm includes the summation operation in (4), which is easily computed numerically. On the contrary, the summation of symbolic expressions is usually complex and does not always yield a compact result. In the obtained solution, for example, it was particularly difficult to calculate coefficient C_3, which contains parameter n in the denominator. Maple operators successfully generalize sequences of polynomials, but do not work with fractional values in symbolic form. Therefore, in [22], a simplified formula was proposed using the mean value theorem, which excludes summation:

$$\omega_*^{-2} = Km\delta_{\max}/2 = m\Delta_{\max}^*.$$

Finding the value of maximum deflection $\delta_{\max}$ among all the deflections due to nodal forces is usually intuitive and reduces to a fairly simple task of identifying the most flexible node. For trussed beams, this is usually the node at the mid-span. In the considered truss, such node can be the one with index $3n + 3$ on the upper chord. The analysis of trusses with a successively increasing number of panels gives the following expressions:

$$\Delta_{\max,1}^* = (c^3+6h^3)/(2h^2EF),$$
$$\Delta_{\max,2}^* = 3(2a^3+2c^3+7h^3)/(h^2EF),$$
$$\Delta_{\max,3}^* = 2(16a^3+11c^3+26h^3)/(h^2EF),$$
$$\Delta_{\max,4}^* = 5(20a^3+12c^3+27h^3)/(h^2EF),$$
$$\Delta_{\max,5}^* = 3(80a^3+45c^3+94h^3)/(h^2EF),...$$

General form of the solution:

$$\Delta_{\max}^* = (D_1a^3+D_2c^3+D_3h^3)/(h^2EF).$$

The coefficients in this expression are obtained by induction:

$$C_1 = n(n-1)(n+1)^2/3,$$
$$C_2 = n(n+1)(n^2+2)/6,$$
$$C_3 = (n+1)(2n^3+4n+3(-1)^n+15)/6.$$

The result is the following formula:

$$\omega_* = h\sqrt{\frac{6EF}{m(n+1)(2n(n^2-1)a^3+n(n^2+2)c^3+(2n^3+4n+3(-1)^n+15)h^3)}}\ . \tag{8}$$

Other versions of the Dunkerley method were considered in [25; 26].

Solution (7) with coefficients (6) and solution (8) can be compared with the numerical one obtained in the Maple system using the Eigenvalues operator from the LinearAlgebra package, designed to calculate the matrix eigenvalues. The algorithm for calculating forces, entering the coordinates of nodes and the order of connections is the same as in the analytical methods.

As an example, a truss with the following characteristics is analyzed: nodal masses $m = 200\,kg$; modulus of elasticity $E = 2.1 \cdot 10^5\,\text{MPa}$; cross-sectional area of the elements $16\,\text{cm}^2$; dimensions $a = 5m$, $h = 1m$. Figure 2 shows the curves of the fundamental frequency versus the number of panels obtained numerically and analytically using the two discussed methods. In the figure, ω_1 denotes the fundamental frequency computed numerically. The highest error of the analytical solutions is observed for lower number of panels. Starting at approximately n = 13, the three methods yield nearly identical frequencies. It is possible to estimate the errors of the methods more accurately and compare them by introducing relative values $\varepsilon_* = |\omega_1 - \omega_*|/\omega_1$ and $\varepsilon_D = |\omega_1 - \omega_D|/\omega_1$ (Figure 3). The figure shows that for n >15 the accuracy of the simplified version of the Dunkerley method (8) is greater than that of the original one (7). The error of the Dunkerley method (when compared with the numerical one) decreases with the increase in the number of panels and reaches a horizontal asymptote of 13 %, which is quite acceptable for using this solution in practice.

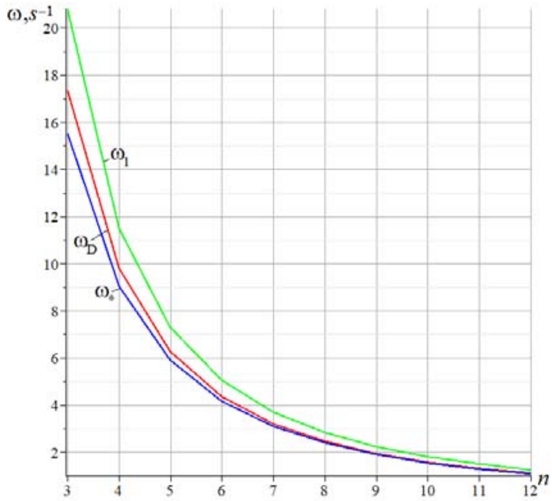

Figure 2. Relationship between the fundamental frequency and the number of panels
S o u r c e : made by the author using the Mapl program

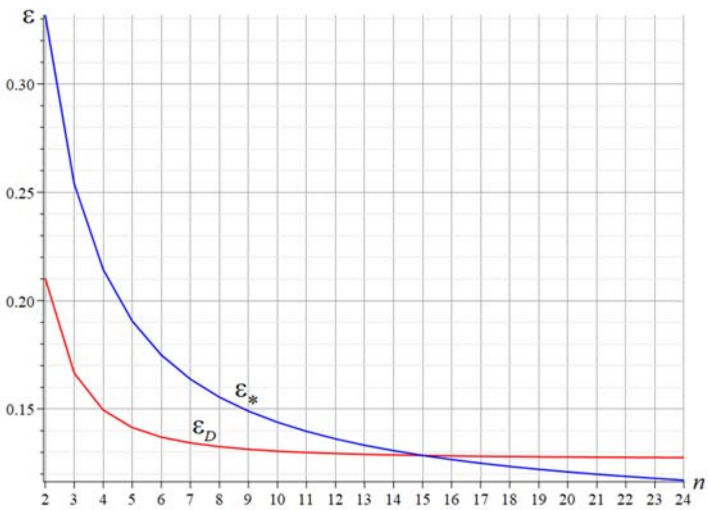

Figure 3. Relationship between the relative error and the number of panels
S o u r c e : made by the author using the Mapl program

5. Conclusion

The model of a plane truss with elastic elements and concentrated nodal masses allows to find an analytical solution for the fundamental natural frequency for an arbitrary number of panels. The solution by the Dunkerley method and its simplified version are comparable in complexity, although the latter solution is still somewhat more compact and in some cases gives a more accurate (under the adopted assumptions) solution. Both methods can be used for a preliminary evaluation of the frequency of natural vibrations of the designed structure. The analytical form of the solutions makes it possible to optimize the structure in terms of choosing the appropriate natural frequency.

The main results:

1. A structure of a statically determinate periodic truss has been developed.
2. Analytical estimates of the fundamental natural frequency are derived as a function of the number of panels.
3. Comparison of the analytical estimates with the numerical solution shows good agreement. The accuracy of estimates increases with the number of panels.
4. For a large number of panels, the simplified Dunkerley method is more accurate than the original method.

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