Mathematical model of deformations of the antenna frame for space communications

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***Abstract* — this research aims to study a novel triangular pyramid-shaped truss with a concave dome that can be utilized as a space communication antenna. The forces in the rods are calculated using the Maple symbolic mathematics system, and the calculation formula coefficients are obtained as general terms of sequences of coefficients in specific solutions. An algorithm is presented to derive the formula for the deflection of the truss under the action of vertical loads. The calculation formula, which is simple and easily verifiable, is valid for any number of panels and can be used for the preliminary evaluation of projected structures. The symbolic form calculations lead to formulas for the forces in the most stretched or compressed rods, and the distribution of forces on the rods is provided. This is the first time that a reliable and easily verifiable formula has been obtained for determining the deflection of a statically determinate truss.**

Keywords — spatial truss, dome, Mohr integral, number of panels, induction, Maple, deflection

I. Introduction

Trusses have many advantages over monolithic or sheet elements in building structures. Most often, girder trusses, arched trusses and frames are used as load-bearing structures. If the structure itself and the load can be decomposed into separate independent flat problems, then the calculation of the spatial structure is reduced to solving several flat problems. At the same time, as a rule, the work of connections is not considered in the work of the entire structure. Most often, calculations of spatial trusses are performed in well-known numerical packages based on the finite element method [1]–[3]. But with a significant number of rods, the numerical calculation can cause some difficulties both at the stage of entering information about the geometry, elastic and strength characteristics of the construction material and in the process of counting and evaluating the result obtained. The main problem of numerical calculation of large-scale structures is the effect of the inevitable accumulation of rounding errors. Therefore, with the development of symbolic transformation systems (Maple, Mathematica, Maxima), the direction of research related to analytical solutions that largely solve these problems began to play an important role. Analytical solutions, in comparison with numerical ones, have such an advantage that mathematical tools can be used for their analysis and optimization, for example, when searching for points of extremes, asymptotes, jumps and inflexion points. This is most effective when applied to solutions covering a wide class of tasks. The more independent parameters describing the geometry, properties of materials and types of loads are included in the calculation formula, the more the analytical result will apply to a wider class of problems. At the same time, with an increase in the number of parameters in the formula, the possibilities of optimization problems are also expanding. In engineering practice, analytical solutions often complement numerical ones [4]. Using the induction method, it is possible to obtain analytical solutions such as solutions in [5], [6], and compact formulas for any order of regularity of the construction [7], [8]. Hutchinson R. G. for the first time considered the problem of finding regular statically determinate core systems [9]. The handbook [10] provide diagrams of a planar regular frame, arch trusses and formulas for both forces in critical rods and deflection and displacement of movable supports. The finite element method in the system of symbolic mathematics in [11], [12] is used to construct algorithms and obtain solutions in an analytical form. There are some different solutions for regular planar trusses [13]–[15], lattice trusses and frames [16]–[18] obtained by induction. The value of these solutions is determined by the number of independent design parameters taken into account in the calculation formulas.

In addition to the dimensions of the structure and the load, the parameters also include integer values that determine the order of its regularity. For trusses, this is usually the number of panels. In beam systems, such a parameter is usually one, in frames — two or three (the number of panels in the crossbar and sides). The most effective method of deriving formulas for deflection and forces in the rods of regular systems is the induction method.

Algorithms for the derivation of formulas for planar trusses are also applicable to spatial trusses. There are known analytical estimates from below of the first natural frequency of free oscillations of trusses [19]–[21] by the Dunkerley method and estimates of the frequency from above by the Rayleigh method using the method of mathematical induction and operators of the Maple system [22]–[24]. By the same method, solutions were obtained for calculating the oscillation frequency of a lattice two-span truss allowing kinematic variability [25].

In this paper, a scheme of a new dome-type spatial structure is proposed and a formula for its deflection is derived by induction (Fig. 1).

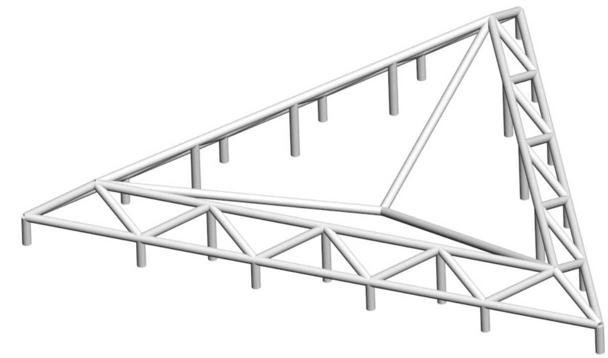

Fig. 1. Construction at $n = 5$.

Unfortunately, the numerical solution has a less applied nature and, often, does not justify the efforts spent on obtaining it, compared with the analytical solution. This is due to the fact that the numerical solution provides us with information about a particular case of truss construction: deflection, distribution of forces in rods and forces in critical rods. The analytical solution has a greater advantage. It, depending on a certain parameter, allows you to obtain a formula generalized for each particular case. Such solutions allow a comprehensive analysis of the truss structure, and choose the most reliable design. After that, if the analytical formula has the necessary accuracy, you can easily substitute the necessary numerical values for the projected structure and get a numerical solution already. The first fairly general formulas for calculating trusses, containing as parameters not only the size and magnitude of the load but also such an ordinal characteristic of regular systems as the number of panels, appeared in the middle of the last century. A number of V.A. Ignatiev's formulas are known [26]. There is also a L.S. Rybakov's algorithm for deriving analytical solutions for rod plane and spatial structures, including complex statically indeterminate [27].

The relevance of the derivation of the formula for the dependence of the deflection of a spatial structure on the number of panels consists of both the need to have simple and reliable test solutions for the calculation and design of structures to evaluate the results obtained in numerical packages and to compare different variants of schemes considered in the design process. Moreover, truss structures are most often used in most antenna frame designs for space communications. In this paper, a spatial construction with three faces is considered.

The novelty of the work lies in the fact that this truss structure and its design scheme are considered for the first time, exactly like the formulas for the analytical evaluation of the deflection along the lower and upper boundaries.

II. Materials and Methods

A. Modeling of design

The structure is supported by vertical posts of height on the lower contour and the upper one. A triangular dome with a height rests on the upper contour and is concave downwards (Fig. 2). The length of the braces connecting the contours is equal to $\sqrt{a^2/3+h^2}$. The truss contains $n_s = 18n-6$ rods, including $6n-3$ vertical support posts.

The nodes of the truss are numbered (Fig. 3). The lattice configuration is entered using specially ordered lists of the numbers of the ends of the corresponding rods, by analogy with the task of graphs in discrete mathematics. The simulation of the trusses scheme begins with entering the coordinates of the nodes and the structure of the rod lattice into the Maple computer mathematics system.

The coordinates of the nodes of the lower (larger) contour have the form:

$$x_i = a(i-1), \quad y_i = 0, \quad z_i = 0,$$

$$x_{i+n} = \frac{a(2n-i+1)}{2}, \quad y_{i+n} = \frac{a\sqrt{3}(i-1)}{2}, \quad z_{i+n} = 0,$$

$$x_{i+2n} = \frac{a(n-i+1)}{2}, \quad y_{i+2n} = \frac{a\sqrt{3}(n-i+1)}{2}, \quad z_{i+2n} = 0,$$

where $i = 1,..,n.$

Coordinates of nodes of the upper (smaller) contour:

$$x_{i+3n} = \frac{a(2i-1)}{2}, \quad y_{i+3n} = \frac{a\sqrt{3}(i-1)}{6}, \quad z_{i+n} = h,$$

$$x_{i+4n-1} = \frac{a(2n-i)}{2}, \quad y_{i+4n-1} = \frac{a\sqrt{3}(3i-2)}{6}, \quad z_{i+4n-1} = h,$$

$$x_{i+5n-2} = \frac{a(n-i+1)}{2}, \quad y_{i+5n-2} = \frac{a\sqrt{3}(3n-3i+1)}{6}, \quad z_{i+5n-2} = h,$$

where $i = 1,..,n-1.$

The top of the dome D has the following coordinates:

$$x_{6n-2} = \frac{na}{2}, \; y_{6n-2} = \frac{y_{3n+1}+y_{4n}+y_{5n-1}}{3}, \; z_{6n-2} = h - \frac{h}{2}(n-1),$$

where $i = 1,..,n-2.$

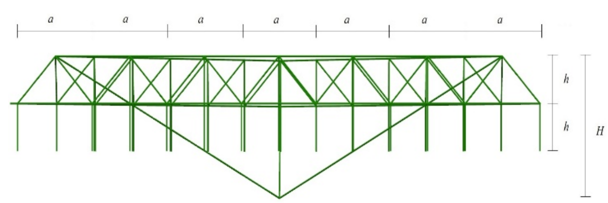

Fig. 2. A view of the triangular pyramid from the side at $n = 7$, $H = h(n-1)/2$.

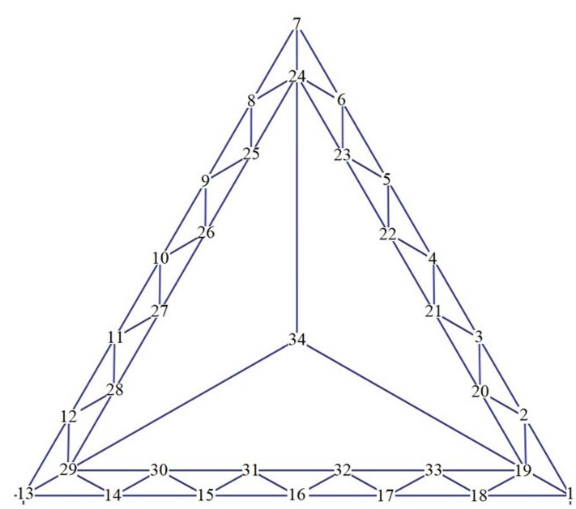

Fig. 3. A view of the structure from above with the numbering of nodes at $n = 6$.

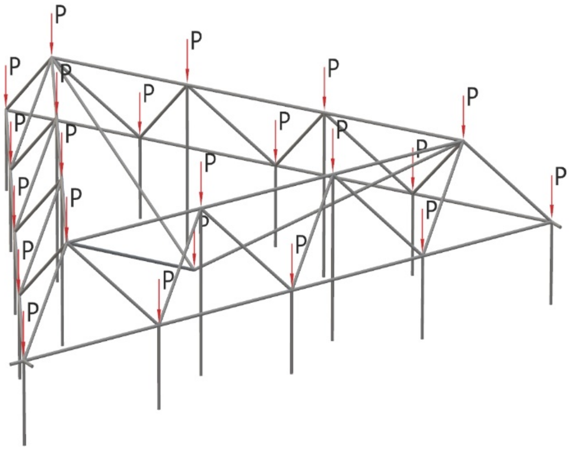

Fig. 4. The effect of the load on the truss belt at $n=4$.

To find the forces in the truss rods, a system of linear equations of all nodes is compiled, excluding nodes fixed on the base, in projection onto three coordinate axes. Into the matrix G the systems include guiding cosines of forces calculated from the coordinates of a three-dimensional grid of nodes. Load data is entered in the right part of the system. In the Maple system, it is most convenient to search for the solution of a system of linear equations composed in matrix form by the inverse matrix method. In the Maple language, it looks the same as when working with numbers. Here is the corresponding fragment of the program:

```
G1:=1/G:        (1)
S:=G1.B:
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where $G1$ is the inverse matrix, S is the vector of unknown forces, B is the vector of the right parts of the system of equations. The dot at the bottom in Maple denotes matrix multiplication or matrix multiplication by a vector.

The Maple system can give a solution in both analytical and numerical form. The distribution of forces on the structure rods under the action of a vertical load uniformly distributed over the nodes is shown in Figure 5. The dimensions of the truss are taken $a=3.0$ m, $h=1.0$ m. The red color indicates tensioned rods with positive forces, blue — compressed (forces less than zero). Thin black segments — the rods are not stressed. These are the rods of the braces in the belt. The thickness of the segments is conditionally proportional to the forces. Force values are calculated at $P=1$. As expected, under such a load (Fig. 4), the rods in the lower chord are stretched, while those in the upper chord are compressed. Significant compressive forces are experienced by three corner rods. Three rods of the inverted dome in the middle of the structure are stretched. Racks (except for corner ones) completely perceive all the force applied to the mount. In all these posts, the compressive force is equal to the force P. The pattern of force distribution in the truss rods is axisymmetric, which is a consequence of the symmetry of the structure and load. The considered algorithm for calculating forces makes it easy to calculate forces under a different load.

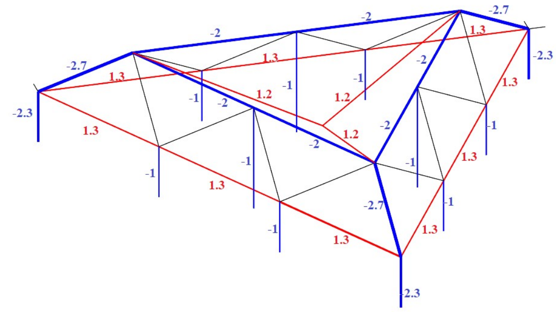

Fig. 5. Distribution of forces on the structure elements, $n=3$.

The deflection of the truss (vertical displacement of the dome hinge) is found using the Mohr integral, which takes into account only the axial forces in the rods:

$$\Delta=\sum_{j=1}^{n_s}\frac{N_j\bar{N}_j l_j}{EF}, \qquad (2)$$

where N_j is the force in the j-th rod of the truss from the applied load, $\bar{N}_j$ is the force in the same rod from a single vertical dimensionless force applied to the node whose displacement is calculated, l_j is the length of the rod, EF is the stiffness of the rods.

B. Analytical solution

Consider the case of a uniformly distributed load of vertical forces P distributed over the structure (Fig. 4). Consistently calculating farms, it can be noted that the type of solution does not depend on the number of panels. Regular systems have this property. The operators of the Maple computer mathematics system allow you to see this and calculate the coefficients at degrees a^3, c^3, d^3, h^3, as well as obtain the following final formula for the dependence of the deflection on the number of panels and the dimensions of the structure:

$$\Delta=P\frac{C_1a^3+C_2c^3+C_3h^3+C_4d^3}{EFh^2}, \qquad (3)$$

where $c=\sqrt{3a^2+9h^2}$, $d=\sqrt{12a^2+9h^2}$, and the coefficients for cubes of sizes form sequences whose common terms can be found using the operators of the Maple system.

The coefficient C_1 at a^3 has the following numerical sequence: $\frac{26}{27},\frac{16}{9},\frac{70}{27},\frac{92}{27},\frac{38}{9},\frac{136}{27},\frac{158}{27},\ldots$ The *rgf_findrecur* operator, which requires an even number of sequence terms, finds a recurrent equation that these numbers satisfy:

$$C_n=2C_{n-1}-C_{n-2}. \qquad (4)$$

The solution of this equation is given by the *rsolve* operator:

$$22n-18. \qquad (5)$$

The sequential solution of recurrent equations for all coefficients gives the following solutions:

$$C_1 = (22n-18)/27,\ C_2 = 4/81,\qquad (6)$$
$$C_3 = 7/3,\ C_4 = (n-1)/162.$$

Compared with known similar solutions, even for planar trusses [28], [29], the solution turned out to be extremely simple. Two coefficients depend linearly on the number of panels, and two are constant. This greatly simplifies calculations and does not require the use of specialized operators rsolve and *rgf_findrecur* of the Maple system.

In the problem under consideration, it was necessary to calculate ten trusses with the number of panels from 2 to 11. Note that symbolic transformations in Maple are performed relatively slowly. The calculation time of the deflection of each subsequent trusses is approximately twice as long as the previous one. Calculating the deflection of trusses with a successively increasing number of panels gives the following formulas:

$$\Delta_2 = P\frac{156a^3+8c^3+378h^3+d^3}{162EFh^2},$$
$$\Delta_3 = P\frac{144a^3+4c^3+189h^3+d^3}{81EFh^2},$$
$$\Delta_4 = P\frac{420a^3+8c^3+378h^3+3d^3}{162EFh^2},$$
$$\Delta_5 = P\frac{276a^3+4c^3+189h^3+2d^3}{81EFh^2},\qquad (7)$$
$$\Delta_6 = P\frac{684a^3+8c^3+378h^3+5d^3}{162EFh^2},$$
$$\Delta_7 = P\frac{408a^3+4c^3+189h^3+3d^3}{81EFh^2},$$
$$\Delta_8 = P\frac{948a^3+8c^3+378h^3+7d^3}{162EFh^2},$$
$$\Delta_9 = P\frac{540a^3+4c^3+189h^3+3d^3}{81EFh^2},$$
$$\ldots$$

C. Numerical solution

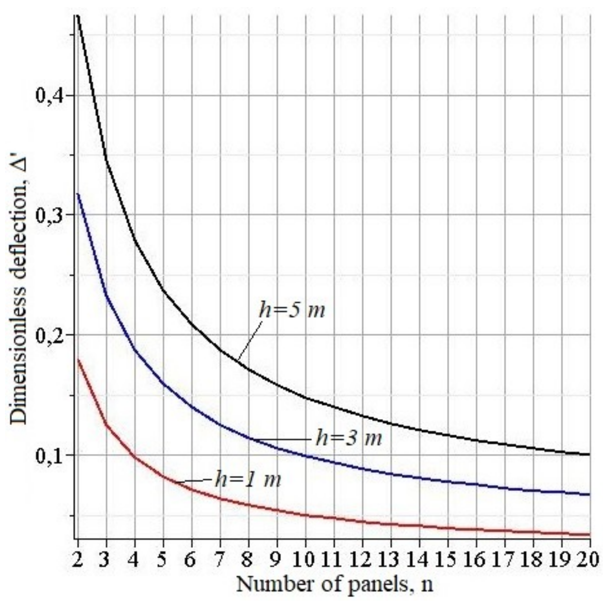

Fig. 6. Dependences of dimensionless deflections at vertex D on the number of panels.

To evaluate the accuracy of an analytical solution, it is necessary to illustrate it numerically. To do this, you need to fix the amount of the total distributed load on the lower belt, depending on the number of panels $P_{sum} = P(2n+1)$. To illustrate, it is necessary to plot the dependence of the dimensionless deflection on the number of panels according to the formula:

$$\Delta' = \frac{\Delta_n EF}{P_{sum}L},\qquad (8)$$

where $L = 4an = 32.0$ m (Fig 6). The asymptote for this solution is horizontal $\lim_{n\to\infty}\Delta_D' = 0$.

Note that the relative deflection decreases with an increase in the number of panels, and the dependence itself becomes hyperbolic.

III. Results and Discussion

In conclusion, the truss structure was modelled using the Maple computer mathematics system, with the coordinates of the nodes, the structure of the rod lattice, and load data entered into the system. The distribution of forces on the rods was calculated and found to be axisymmetric. The deflection of the truss was calculated using the Mohr integral, taking into account only the axial forces in the rods. An analytical solution was obtained for the dependence of the deflection on the number of panels and dimensions of the structure. The solution was found to be simple, with two coefficients depending linearly on the number of panels and two being constant, which greatly simplifies the calculations. The calculations were performed for ten trusses with the number of panels ranging from 2 to 11.

The parameter of this type of statically determinate dome-type spatial truss is the number of panels that allows you to use the inductive method to obtain basic formulas for estimating structural deformations. These estimates are convenient to use in optimization problems and as test estimates for numerical solutions [30], [31]. The proposed trusses scheme is new, and analytical solutions for its deflection have been obtained for the first time.

Technical difficulties had to be overcome in the process of outputting formulas in the symbolic transformation system, which works much slower than packages based on numerical methods. With an increase in the number of panels on the trusses, the calculation time increased dramatically, so either a processor with good characteristics or a significant counting time was required. Induction is carried out by one parameter per step and some time is spent on each step so the total time spent will be too large. The solution to this problem was somewhat simplified in cases where individual coefficients did not depend on the number of panels.

The forces in the coating rods are found by cutting nodes in symbolic form using operators of the Maple symbolic mathematics system. The coefficients of the calculation formula are calculated as general terms of sequences of coefficients in particular solutions. Based on calculations in symbolic form, formulas for the forces in the most stretched or compressed rods are obtained. For example, the distribution of forces on the rods of the structure is given.

IV. Conclusion

A new scheme of a spatial statically determinate pyramidal truss with two support contours is proposed. The proposed trusses scheme is new, and analytical solutions for its deflection have been obtained for the first time.

Such a design scheme can be used in antenna designs for space communications, for which rigidity requirements are of particular importance. Simple and easily verifiable calculation formulas, valid for any number of panels, provide reliable verification of numerical solutions and are applicable for preliminary evaluation of projected structures. The resulting solution can also be used as a basic scheme for solving a statically indeterminate problem for a more complex structure with additional connections, for example, when attaching the top of the dome to all nodes on the belt of the structure. For this purpose, based on calculations in symbolic form, formulas for forces in the most stretched or compressed rods are obtained and the distribution of forces on the rods of the structure is given.

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